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SOBRE GENERALIZACIONES DE LA TEORÍA DE FREDHOLM, TEOREMAS TIPO WEYL, ESTRUCTURAS MINIMALES Y TOPOLOGÍAS GENERALIZADAS

Presentado por:
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Trabajo de ascenso presentado como requisito parcial para ascender a la categoría de Profesor Titular

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A mi madre María Eugenia Carpintero,
y a mis hijas:
María Laura y María de los Angeles

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A Dios y la Virgen del Valle.
A mis Maestros,
por todo lo que de ellos aprendí

ÍNDICE

INTRODUCCIÓN

En este trabajo se presentan algunos resultados de las investigaciones que el autor ha venido desarrollando, junto a otros investigadores, durante estos últimos cuatro años en las áreas de teoría de operadores, teoría espectral y topología, y que han conducido a varios artículos de investigación con inéditos resultados, publicados o aceptados para su publicación en revistas arbitradas nacionales e internacionales, así como también a un buen número de tesis de pregrado y de postgrado.

El trabajo presenta dos partes claramente diferenciadas, la primera parte trata de la teoría de Fredholm y su generalización en el sentido de Berkani, estudiada desde el punto de vista de la teoría espectral local, y sus implicaciones en los teoremas generalizados de Weyl. Esta parte consta de los siguientes artículos:

> P. Aiena, C. Carpintero y E. Rosas, "Browder's theorems and the spectral mapping theorem". Divulgaciones Matemáticas, Vol. 15, N^0. 2 (2007).
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> P. Aiena, M. Biondi y C. Carpintero, "On Drazin Invertibility". Proceedings of the American Mathematical Society, Vol. 136, N^0. 8 (2008).
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> C. Carpintero, O. Garcia, E. Rosas y J. Sanabria, "B-Browder spectra and localized SVEP". Rendiconti del Circolo Matematico di Palermo, 57, N^0. 2 (2008).
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> C. Carpintero, D. Muñoz, E. Rosas, O. Garcia y J. Sanabria, "Generalized Weyl's theorems for polaroid operators ". Por aparecer en Carpathian Journal of Mathematics, Vol. 27, N^0. 1 (2011).

En los cuales se producen aportes significativos en lo referente a relaciones espectrales entre los espectros originados en la teoría de Fredholm y los originados en la generalización de esta en sentido de Berkani, así como también, se extienden y mejoran muchos resultados aparecidos recientemente en la literatura relacionada con estas áreas.

La segunda parte, trata de nociones topológicas generalizadas vía operadores y estructuras minimales, y consta de los artículos siguientes:

C. Carpintero, E. Rosas y J. Sanabria, "A Tychonoff theorem for α-compactness and some applications". Indian Journal of Mathematics, Vol. 49, N^0. 1 (2007).

C. Carpintero, E. Rosas y M. Salas, "Minimal structures and separations properties". International Journal of Pure and Applied Mathematics, Vol. 34, N^0. 4 (2007).

M. Salas, C. Carpintero y E. Rosas, "Conjuntos m_X-cerrados generalizados". Divulgaciones Matemáticas, Vol. 15, N^0. 1 (2007).

C. Carpintero, E. Rosas, O. Özbakir y J. Salazar, "Inadmissible families and product of generalized topologies". International Mathematical Forum, Vol. 5, N^0. 63 (2010).

En dichos articulos, se logran probar nuevos resultados relativos a formas generalizadas de compacidad sobre el espacio producto cuando actúa un operador. Además, se extienden propiedades de separación y de conjuntos g-cerrados en el contexto de las m-estructuras, y en esa misma dirección se obtienen resultados sobre el producto de m-espacios y topologías generalizadas, respectivamente, de las cuales se derivan importantes aplicaciones concretas.

PARTE I

TEORÍA DE FREDHOLM, SU GENERALIZACIÓN Y TEOREMAS TIPO WEYL

Con el fin de dar una visión global del contenido, orientación y propósito de los artículos tratados en esta parte del trabajo, se describen a continuación los aspectos que motivaron el desarrollo de los mismos, así como también los resultados obtenidos.

La teoría de Fredholm tuvo su origen en el estudio de la solución de las ecuaciones integrales desde un punto de vista abstracto. Ha sido aplicada en teoría de espacios de Banach y ha sido una de las fuentes de inspiración para el estudio de diversas nociones entre las que podemos citar; semigrupos de operadores, operadores estrictamente singulares, estrictamente cosingulares, etc. Ciertos tipos especiales de operadores juegan un importante papel en dicha teoría, estos son los operadores de Browder (conocidos también en la literatura como operadores de Riesz-Schauder), sus generalizaciones, que son los operadores superiormente o inferiormente semi-Browder (introducidos por R. Harte ([12])) y los operadores de Weyl, denominados así en honor a Herman Weyl ([20]). Inspirado en el trabajo de Weyl ([20]), Coburn ([6]) introduce en forma abstracta el teorema de Weyl. Despúes de Coburn, diversos autores empleando los espectros derivados de la teoría clásica de Fredholm, introducen otras variantes del teorema de Weyl. Entre los que podemos citar, el teorema de Browder y el teorema de a-Browder, introducidos por R. Harte y W. Y. Lee en 1997 ([13]), y el teorema de a-Weyl, introducido por V. Rakočević ([19]).

Clásicamente, la teoría de Fredholm ha sido descrita desde varios puntos de vista, a saber, a través de caracterizaciones perturbativas o mediante ideales ([14],[5]). Pero, en el año 1975, J. Finch ([10]) introduce una versión localizada en un punto de la propiedad de la extensión univaluada para un operador, $SVEP$ (por sus siglas en inglés: single valued extension property), la cual resultó ser muy apropiada para el estudio de propiedades espectrales relativas a la teoría de Fredholm, ya que permite estudiar los operadores semi-Fredholm empleando técnicas y elementos propios de la teoría espectral local. Esta fusión de la teoría de Fredholm y la teoría espectral local, a través de la $SVEP$ local, ha producido una gran cantidad de resultados en esta última década. En esta parte del trabajo se presentan algunos resultados en esta dirección, en los que el autor ha estado trabajando en estos últimos cuatro años. En primer lugar se presenta el artículo titulado "Browder's theorems and the spectral mapping theorem" de P. Aiena, C. Carpintero y E. Rosas, publicado el año 2007, en la revista Divulgaciones Matemáticas, en el cual se dan caracterizaciones para los teoremas de Browder y a-Browder de un operador en términos de la $SVEP$ local y de ciertos subconjuntos propios del plano complejo asociados con un operador.

Motivado por la teoría de Fredholm, M. Berkani ([2],[3]) introduce varias clases de operadores mucho más amplias que la clase de los operadores semi-Fredholm, los cuales son denominados operadores semi B-Fredholm, semi B-Browder y semi B-Weyl. Asi mismo, junto con J. Koliha ([4]) introducen, en el contexto de los espectros generalizados, nuevas versiones más fuertes que los teoremas de Browder, a-Browder y a-Weyl, conocidos como los teoremas generalizados de Berkani. Estos son, los teoremas de B-Browder, a-B-Browder, B-Weyl y a-B-Weyl. En este contexto se presenta, en segundo lugar, el artículo titulado "On Drazin Invertibility" de P. Aiena, M. Biondi y C. Carpintero, publicado el año 2008, en la revista Proceedings of the American Mathematical Society, en el que se aborda el estudio de una particularización del concepto de Drazin invertibilidad, dado por M. P. Drazin ([9]), para el caso del algebra de los operadores que actúan de un espacio de Banach complejo en si mismo; obteniendo caracterizaciones de los operadores que son Drazin invertibles, a través de la $SVEP$, así como algunas relaciones de éstos con los operadores generalizados de Berkani. Seguidamente se presenta también, el artículo titulado "B-Browder spectra and localized SVEP" de C. Carpintero, O. Garcia, E. Rosas y J. Sanabria, publicado el año 2008, en la revista Rendiconti del Circolo Matematico di Palermo, en el cual se logra extender los resultados relativos a los espectros semi-Browder obtenidos en ([1]) para esta nueva clase de espectros semi B-Browder, semi B-Weyl y semi B-Fredholm, introducidos por Berkani.

Finalmente, se presenta el artículo titulado "Generalized Weyl's theorems for polaroid operators " de C. Carpintero, D. Muñoz, E. Rosas, O. Garcia y J. Sanabria; el cual aparecerá publicado este año 2011, en la revista Carpathian

Journal of Mathematics. En este artículo, se logra obtener una multiplicidad de descripciones tanto de tipo algebraico, como topológicas, para los teoremas de B-Weyl y a-B-Weyl de un operador, en base a las cuales se obtienen descripciones de los teoremas de B-Weyl y a-B-Weyl para la importante clase de los operadores polaroides. De esta forma se obtiene, de manera inmediata, la caracterización de estos teoremas para distintas subclases particulares de operadores contenidas en esta, tales como los operadores paranormales, algebraicamente paranormales, hiponormales, M-hiponormales, p-hiponormales y log-hiponormales, entre otros, cada uno de los cuales hasta ahora habían sido estudiados en forma aislada y con argumentos mucho más sofisticados.

1.1. TEOREMAS DE BROWDER Y EL TEOREMA DE LA APLICACIÓN ESPECTRAL

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Browder's theorems and the spectral mapping theorem

Los teoremas de Browder y el teorema de la aplicación espectral

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Abstract

A bounded linear operator $T \in L(X)$ on a Banach space X is said to satisfy Browder's theorem if two important spectra, originating from Fredholm theory, the Browder spectrum and the Weyl spectrum, coincide. This expository article also concerns with an approximate point version of Browder's theorem. A bounded linear operator $T \in L(X)$ is said to satisfy a-Browder's theorem if the upper semi-Browder spectrum coincides with the approximate point Weyl spectrum. In this note we give several characterizations of operators satisfying these theorems. Most of these characterizations are obtained by using a localized version of the single-valued extension property of T. This paper also deals with the relationships between Browder's theorem, a-Browder's theorem and the spectral mapping theorem for certain parts of the spectrum.

Key words and phrases: Local spectral theory, Fredholm theory, Weyl's theorem.

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Resumen

Un operador lineal acotado $T \in L(X)$ sobre un espacio de Banach X se dice que satisface el teorema de Browder, si dos importantes espectros, en el contexto de la teoría de Fredholm, el espectro de Browder y el espectro de Weyl, coinciden. Este artículo expositivo trata con una versión puntual del teorema de Browder. Un operador lineal acotado $T \in L(X)$ sobre un espacio de Banach X se dice que satisface el teorema de a-Browder si el espectro superior semi-Browder coincide con el espectro puntual aproximado de Weyl. En este nota damos varias caracterizaciones de operadores que satisfacen estos teoremas. La mayorí de estas caracterizaciones se obtienen de versiones localizadas de la propiedad de extensión univaluada de T. Este trabajo también considera las relaciones entre el teorema de Browder el teorema a-Browder y el teorema de transformación espectral para ciertas partes del espectro.
Palabras y frases clave: Teoría espectral local, teoría de Fredholm, teorema de Weyl.

1 Introduction and definitions

If X is an infinite-dimensional complex Banach space and $T \in L(X)$ is a bounded linear operator, we denote by $\alpha(T) := \dim \ker T$, the dimension of the null space $\ker T$, and by $\beta(T) := \operatorname{codim} T(X)$ the codimension of the range $T(X)$. Two important classes in Fredholm theory are given by the class of all *upper semi-Fredholm* operators $\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty$ and $T(X)$ is closed$\}$, and the class of all *lower semi-Fredholm* operators defined by $\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$. The class of all *semi-Fredholm operators* is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ defines the class of all *Fredholm operators.* The *index* of $T \in \Phi_\pm(X)$ is defined by ind $(T) := \alpha(T) - \beta(T)$. Recall that a bounded operator T is said *bounded below* if it is injective and it has closed range. Define

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \le 0\},$$

and

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \ge 0\}.$$

The set of *Weyl operators* is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X) : \operatorname{ind} T = 0\}.$$

The classes of operators defined above generate the following spectra. The *Fredholm spectrum* (known in literature also as *essential spectrum*) is defined by

$$\sigma_{\mathrm{f}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

The *Weyl spectrum* is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{\mathrm{wa}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{\mathrm{ws}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

Denote by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

the *approximate point spectrum*, and by

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\},$$

the *surjectivity spectrum*.

The spectrum $\sigma_{\mathrm{wa}}(T)$ admits a nice characterization: it is the intersection of all approximate point spectra $\sigma_{\mathrm{a}}(T+K)$ of compact perturbations K of T, while, dually, $\sigma_{ws}(T)$ is the intersection of all surjectivity spectra $\sigma_{\mathrm{s}}(T+K)$ of compact perturbations K of T, see for instance [1, Theorem 3.65]. From the classical Fredholm theory we have

$$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ws}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{wa}}(T^*) = \sigma_{\mathrm{ws}}(T).$$

This paper concerns also with two other classical quantities associated with an operator T: the *ascent* $p := p(T)$, i.e. the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$, and the *descent* $q := q(T)$, i.e the smallest non-negative integer q, such that $T^q(X) = T^{q+1}(X)$. If such integers do not exist we shall set $p(T) = \infty$ and $q(T) = \infty$, respectively. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ belongs to the spectrum $\sigma(T)$ and is a pole of the function resolvent $\lambda \to (\lambda I - T)^{-1}$, see Proposition 50.2 of [18]. The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty\},$$

the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

while the class of all *lower semi-Browder operators* is defined

$$B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

Obviously, $B(X) = B_+(X) \cap B_-(X)$ and

$$B(X) \subseteq W(X), \quad B_+(X) \subseteq W_+(X), \quad B_-(X) \subseteq W_-(X)$$

see [1, Theorem 3.4].

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\},$$

and analogously the *lower semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_-(X)\}.$$

Clearly,

$$\sigma_{\mathrm{f}}(T) \subseteq \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T),$$

and

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{lb}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*).$$

Furthermore, by part (v) of Theorem 3.65 [1] we have

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T) \cup \operatorname{acc} \sigma_{\mathrm{a}}(T), \tag{1}$$

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T) \cup \operatorname{acc} \sigma_{\mathrm{s}}(T), \tag{2}$$

and

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc} \sigma(T), \tag{3}$$

where we write acc K for the set of all cluster points of $K \subseteq \mathbb{C}$.

A bounded operator $T \in L(X)$ is said to be *semi-regular* if it has closed range and

$$\ker T^n \subseteq T(X) \quad \text{for all } n \in \mathbb{N}.$$

The *Kato spectrum* is defined by

$$\sigma_{\mathrm{k}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

Note that $\sigma_{\mathrm{k}}(T)$ is a non-empty compact subset of $\mathbb{C}$, since it contains the boundary of the spectrum, see [1, Theorem 1.75]. An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T\mid M$ is semi-regular and $T\mid N$ is quasi-nilpotent. A relevant case is obtained if we assume in the definition above that $T\mid N$ is nilpotent. In this case T is said to be of *Kato type*. If N is finite-dimensional then T is said to be *essentially semi-regular*. Every semi-Fredholm operator is essentially semi-regular, by the classical result of Kato, see Theorem 1.62 of [1]. Recall that $T \in L(X)$ is said to admit a *generalized inverse* $S \in L(X)$ if TST=T. It is well known that T admits a generalized inverse if and only if both subspaces $\ker T$ and $T(X)$ are complemented in X. It is well-known that every Fredholm operator admits a generalized inverse, see Theorem 7.3 of [1]. A "complemented" version of Kato operators is given by the *Saphar operators*: $T \in L(X)$ is said to be Saphar if T is semi-regular and admits a generalized inverse. The *Saphar spectrum* is defined by

$$\sigma_{\mathrm{sa}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Saphar}\}.$$

Clearly, $\sigma_{\mathrm{k}}(T) \subseteq \sigma_{\mathrm{sa}}(T)$, so $\sigma_{\mathrm{sa}}(T)$ is nonempty; for other properties on Saphar operators see Müller [22, Chapter II, §13].

2 SVEP

There is an elegant interplay between Fredholm theory and the single-valued extension property, an important role that has a crucial role in local spectral theory. This property was introduced in the early years of local spectral theory by Dunford [13], [14] and plays an important role in the recent monographs by Laursen and Neumann [20], or by Aiena [1]. Recently, there has been a flurry of activity regarding a localized version of the single-valued extension property, considered first by [15] and examined in several more recent papers, for instance [21], [5], and [7].

Definition 2.1. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single valued extension property *at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic* *function* $f : U \to X$ *which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0, \quad \textit{for all } \lambda \in U$$

is the function $f \equiv 0$. *An operator* $T \in L(X)$ *is said to have SVEP if* T *has SVEP at every point* $\lambda \in \mathbb{C}$.

The SVEP may be characterized by means of some typical tools of the local spectral theory, see [8] or Proposition 1.2.16 of [20]. Note that by the identity theorem for analytic function both T and T^* have SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, both T and the dual T^* have SVEP at the isolated points of $\sigma(T)$.

A basic result links the ascent, descent and localized SVEP:

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda,$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda,$$

see [1, Theorem 3.8].

Furthermore, from the definition of localized SVEP it is easy to see that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{4}$$

while

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda.$$

An important subspace in local spectral theory is the *quasi-nilpotent part* of T, namely, the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, ker $(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$. Moreover, T is quasi-nilpotent if and only if $H_0(T) = X$, see [1, Theorem 1.68]. If $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n\|x\|$ for all $n \in \mathbb{N}$, see [1] for informations on the subspaces $H_0(T)$, $K(T)$. The subspaces $H_0(T)$ and $K(T)$ are invariant under T and may be not closed. We have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda,$$

see [5].

In the following theorem we collect some characterizations of SVEP for operators of Kato type.

Theorem 2.2. *Suppose that $\lambda_0 I - T$ is of Kato type. Then the following statements are equivalent:*

(i) *T has SVEP at λ_0;*

(ii) *$p(\lambda_0 I - T) < \infty$;*

(iii) *$H_0(\lambda_0 I - T)$ is closed;*

(iv) *$\sigma_{\mathrm{a}}(T)$ does not cluster at λ.*

If $\lambda_0 I - T$ is essentially semi-regular the statements (i) - (iv) *are equivalent to the following condition:*

(v) *$H_0(\lambda_0 I - T)$ is finite-dimensional.*

If $\lambda_0 I - T$ is semi-regular the statements (i) - (v) *are equivalent to the following condition:*

(vi) *$\lambda_0 I - T$ is injective.*

Dually, the following statements are equivalent:

(vii) *T^* has SVEP at λ_0;*

(viii) *$q(\lambda_0 I - T) < \infty$;*

(ix) *$\sigma_{\mathrm{s}}(T)$ does not cluster at λ.*

If $\lambda_0 I - T$ is essentially semi-regular the statements (vi) - (viii) *are equivalent to the following condition:*

(x) *$K(\lambda I - T)$ is finite-codimensional.*

If $\lambda_0 I - T$ is semi-regular the statements (vii) - (x) *are equivalent to the following condition:*

(xi) *$\lambda_0 I - T$ is surjective.*

Remark 2.3. Note that the condition $p(T) < \infty$ (respectively, $q(T) < \infty$) implies for a semi-Fredholm that $\operatorname{ind} T \leq 0$ (respectively, $\operatorname{ind} T \geq 0$), see [1, Theorem 3.4]. Consequently, if T has SVEP then $\lambda \notin \sigma_{\mathrm{f}}(T)$ then $\operatorname{ind}(\lambda I - T) \leq 0$, while if T^* has SVEP then $\operatorname{ind}(\lambda I - T) \geq 0$.

Let λ_0 be an isolated point of $\sigma(T)$ and let P_0 denote the spectral projection

$$P_0 := \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} \, \mathrm{d}\lambda$$

associated with $\{\lambda_0\}$, via the classical Riesz functional calculus. A classical result shows that the range $P_0(X)$ is $N := H_0(\lambda_0 I - T)$, see Heuser [18, Proposition 49.1], while $\ker P_0$ is the analytic core $M := K(\lambda_0 I - T)$ of $\lambda_0 I - T$, see [24] and [21]. In this case, $X = M \oplus N$ and

$$\sigma(\lambda_0 I - T|N) = \{\lambda_0\}, \quad \sigma(\lambda_0 I - T|M) = \sigma(T) \setminus \{\lambda_0\},$$

so $\lambda_0 I - T|M$ is invertible and hence $H_0(\lambda_0 I - T|M) = \{0\}$. Therefore from the decomposition $H_0(\lambda_0 I - T) = H_0(\lambda_0 I - T|M) \oplus H_0(\lambda_0 I - T|N)$ we deduce that $N = H_0(\lambda_0 I - T|N)$, so $\lambda_0 I - T|N$ is quasi-nilpotent. Hence the pair (M, N) is a GKD for $\lambda_0 I - T$.

Corollary 2.4. *Let λ_0 be an isolated point of $\sigma(T)$. Then*

$$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$$

and the following assertions are equivalent:

(i) $\lambda_0 I - T$ *is semi-Fredholm;*

(ii) $H_0(\lambda_0 I - T)$ *is finite-dimensional;*

(iii) $K(\lambda_0 I - T)$ *is finite-codimensional.*

Proof. Since for every operator $T \in L(X)$, both T and T^* have SVEP at any isolated point, the equivalence of the assertions easily follows from the decomposition $X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$, and from Theorem 2.2. ∎

3 Browder's theorem

In 1997 Harte and W. Y. Lee [16] have christened that *Browder's theorem* holds for T if

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T),$$

or equivalently, by (3), if

$$\operatorname{acc}\sigma(T) \subseteq \sigma_{\mathrm{w}}(T). \tag{5}$$

Let write *iso* K for the set of all isolated points of $K \subseteq \mathbb{C}$. To look more closely to Browder's theorem, let us introduce the following parts of the spectrum: For a bounded operator $T \in L(X)$ define

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \in \mathcal{B}(X)\},$$

the set of all *Riesz points* in $\sigma(T)$. Finally, let us consider the following set:

$$\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T).$$

Clearly, if $\lambda \in \Delta(T)$ then $\lambda I - T \in W(X)$ and since $\lambda \in \sigma(T)$ it follows that $\alpha(\lambda I - T) = \beta(\lambda I - T) > 0$, so we can write

$$\Delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W(X),\, 0 < \alpha(\lambda I - T)\}.$$

The set $\Delta(T)$ has been recently studied in [16], where the points of $\Delta(T)$ are called *generalized Riesz points*. It is easily seen that

$$p_{00}(T) \subseteq \Delta(T) \quad \text{for all } T \in L(X).$$

Our first result shows that Browder's theorem is equivalent to the localized SVEP at some points of $\mathbb{C}$.

Theorem 3.1. *For an operator $T \in L(X)$ the following statements are equivalent:*

(i) $p_{00}(T) = \Delta(T)$;

(ii) *T satisfies Browder's theorem;*

(iii) *T^* satisfies Browder's theorem;*

(iv) *T has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$;*

(v) *T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$.*

From Theorem 3.1 we deduce that the SVEP for either T or T^* entails that both T and T^* satisfy Browder's theorem. However, the following example shows that SVEP for T or T^* is a not necessary condition for Browder's theorem.

Example 3.2. Let $T := L \oplus L^* \oplus Q$, where L is the unilateral left shift on $\ell^2(\mathbb{N})$, defined by

$$L(x_1, x_2, \dots) := (x_2, x_3, \cdots), \quad (x_n) \in \ell^2(\mathbb{N}),$$

and Q is any quasi-nilpotent operator. L does not have SVEP, see [1, p. 71], so also T and T^* do not have SVEP, see Theorem 2.9 of [1]. On the other hand, we have $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \mathbf{D}$, where $\mathbf{D}$ is the closed unit disc in $\mathbb{C}$, so that Browder' theorem holds for T.

A very clear spectral picture of operators for which Browder's theorem holds is given by the following theorem:

Theorem 3.3. *[3] For an operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies Browder's theorem;*

(ii) *Every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$;*

(iii) *$\Delta(T) \subseteq \partial\sigma(T)$, $\partial\sigma T)$ the topological boundary of $\sigma(T)$;*

(iv) $\mathrm{int}\,\Delta(T) = \emptyset$, *$int\,\Delta(T)$ the interior of $\Delta(T)$;*

(v) $\sigma(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{iso} \sigma(T)$.

(vi $\Delta(T) \subseteq \sigma_{\mathrm{k}}(T)$*;*

(vii) $\Delta(T) \subseteq \operatorname{iso} \sigma_{\mathrm{k}}(T)$*;*

(viii) $\Delta(T) \subseteq \sigma_{\mathrm{sa}}(T)$*;*

(ix) $\Delta(T) \subseteq \operatorname{iso} \sigma_{\mathrm{sa}}(T)$.

Other characterizations of Browder's theorem involve the quasi-nilpotent part and the analytic core of T:

Theorem 3.4. *For a bounded operator $T \in L(X)$ Browder's theorem holds precisely when one of the following statements holds;*

(i) $H_0(\lambda I - T)$ *is finite-dimensional for every* $\lambda \in \Delta(T)$*;*

(ii) $H_0(\lambda I - T)$ *is closed for all* $\lambda \in \Delta(T)$*;*

(iii) $K(\lambda I - T)$ *is finite-codimensional for all* $\lambda \in \Delta(T)$.

Define

$$\sigma_1(T) := \sigma_{\mathrm{w}}(T) \cup \sigma_{\mathrm{k}}(T).$$

We show now, by using different methods, some recent results of X. Cao, M. Guo, B. Meng [10]. These results characterize Browder's theorem through some special parts of the spectrum defined by means the concept of semi-regularity.

Theorem 3.5. *For a bounded operator the following statements are equivalent:*

(i) T *satisfies Browder's theorem;*

(ii) $\sigma(T) = \sigma_1(T)$*;*

(iii) $\Delta(T) \subseteq \sigma_1(T)$*,*

(iv) $\Delta(T) \subseteq \operatorname{iso} \sigma_1(T)$.

(v) $\sigma_{\mathrm{b}}(T) \subseteq \sigma_1(T)$*;*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has been proved in [10], but is clear from Theorem 3.3.

(i) $\Leftrightarrow$ (iii) Suppose that T satisfies Browder's theorem or equivalently, by Theorem 3.3, that $\Delta(T) \subseteq \sigma_{\mathrm{k}}(T)$. Then $\Delta(T) \subseteq \sigma_{\mathrm{w}}(T) \cup \sigma_{\mathrm{k}}(T) = \sigma_1(T)$. Conversely, if $\Delta(T) \subseteq \sigma_1(T)$ then $\Delta(T) \subseteq \sigma_{\mathrm{k}}(T)$, since by definition $\Delta(T) \cap \sigma_{\mathrm{w}}(T) = \emptyset$.

(iii)$\Rightarrow$ (iv) Suppose that the inclusion $\Delta(T) \subseteq \sigma_1(T)$ holds. We know by the first part of the proof that this inclusion is equivalent to Browder's theorem, or also to the equality $\sigma(T) = \sigma_1(T)$. By Theorem 3.3 we then have

$$\Delta(T) \subseteq \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma_1(T).$$

(iv)$\Rightarrow$ (iii) Obvious.

(i) $\Rightarrow$ (v) If T satisfies Browder's theorem then $\sigma_{\rm b}(T) = \sigma_{\rm w}(T) \subseteq \sigma_1(T)$.

(v) $\Rightarrow$ (ii) Suppose that $\sigma_{\rm b}(T) \subseteq \sigma_1(T)$. We show that $\sigma(T) = \sigma_1(T)$. It suffices only to show $\sigma(T) \subseteq \sigma_1(T)$. Let $\lambda \notin \sigma_1(T) = \sigma_{\rm w}(T) \cup \sigma_{\rm k}(T)$. Then $\lambda \notin \sigma_{\rm b}(T)$, so λ is an isolated point of $\sigma(T)$ and $\alpha(\lambda I - T) = \beta(\lambda I - T)$. Since $\lambda \notin \sigma_{\rm k}(T)$ then $\lambda I - T$ is semi-regular and the SVEP ar λ implies by Theorem 2.2 that $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, i.e. $\lambda \notin \sigma(T)$. ∎

By passing we note that the paper by X. Cao, M. Guo, and B. Meng [10] contains two mistakes. The authors claim in Lemma 1.1 that $\operatorname{iso}\sigma_{\rm k}(T) \subseteq \sigma_{\rm w}(T)$ for every $T \in L(X)$. This is false, for instance if λ is a Riesz point of T then $\lambda \in \partial\sigma(T)$, since λ is isolated in $\sigma(T)$, and hence $\lambda \in \sigma_{\rm k}(T)$, see [1, Theorem 1.75], so $\lambda \in \operatorname{iso}\sigma_{\rm k}(T)$. On the other hand, $\lambda I - T$ is Weyl and hence $\lambda \notin \sigma_{\rm w}(T)$.
Also the equivalence: Browder's theorem for $T \Leftrightarrow \sigma(T) \setminus \sigma_{\rm k}(T) \subseteq \operatorname{iso}\sigma_{\rm k}(T)$, claimed in Corollary 2.3 of [10] is not corrected, the correct statement is the equivalence (i) $\Leftrightarrow$ (vi) established in Theorem 3.3.

Denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood of $\sigma(T)$, let $f(T)$ be defined by means of the classical functional calculus. It should be noted that the spectral mapping theorem does not hold for $\sigma_1(T)$. In fact we have the following result.

Theorem 3.6. *[10] Suppose that $T \in L(X)$. For every $f \in \mathcal{H}(\sigma(T))$ we have $\sigma_1(f(T)) \subseteq f(\sigma_1(T))$. The equality $f(\sigma_1(T)) = \sigma_1(f(T))$ holds for every $f \in \mathcal{H}(\sigma(T))$ precisely when the spectral mapping theorem holds for $\sigma_{\rm w}(T)$, i.e.,*

$$f(\sigma_{\rm w}(T)) = \sigma_{\rm w}(f(T)) \quad \text{for all } f \in \mathcal{H}(\sigma(T)).$$

Note that the spectral mapping theorem for $\sigma_{\rm w}(T)$ holds if either T or T^* satisfies SVEP, see also next Theorem 4.3. This is also an easy consequence of Remark 2.3.

Theorem 3.7. *[10] The spectral mapping theorem holds for $\sigma_1(T)$ precisely when $ind\,(\lambda I - T) \cdot ind\,(\mu I - T) \geq 0$ for each pair $\lambda, \mu \notin \sigma_{\rm f}(T)$.*

In general, Browder's theorem for T does not entail Browder's theorem for $f(T)$. However, we have the following result.

Theorem 3.8. *Suppose that both $T \in L(X)$ and $S \in L(X)$ satisfy Browder's theorem, $f \in \mathcal{H}(\sigma(T))$ and p a polynomial. Then we have:*

(i) *[10] Browder's theorem holds for $f(T)$ if and only if $f(\sigma_1(T)) = \sigma_1(f(T))$.*

(ii) *[10] Browder's theorem holds for $T \oplus S$ if and only if $\sigma_1(T) \cup \sigma_1(S) = \sigma_1(T \oplus S)$.*

(iii) *[16] Browder's theorem holds for $p(T)$ if and only if $p(\sigma_{\mathrm{w}}(T)) \subseteq \sigma_{\mathrm{w}}(p(T))$.*

(iv) *[16] Browder's theorem holds for $T \oplus S$ if and only if $\sigma_{\mathrm{w}}(T) \cup \sigma_{\mathrm{w}}(S) \subseteq \sigma_{\mathrm{w}}(T \oplus S)$.*

Browder's theorem survives under perturbation of compact operators K commuting with T. In fact, we have

$$\sigma_{\mathrm{w}}(T+K) = \sigma_{\mathrm{w}}(T) \quad \text{and} \quad \sigma_{\mathrm{b}}(T+K) = \sigma_{\mathrm{b}}(T); \tag{6}$$

the first equality is a standard result from Fredholm theory, while the second equality is due to V. Rakočević [23]. It is not difficult to extend this result to Riesz operators commuting with T (recall that $K \in L(X)$ is said to be a *Riesz operator* if $\lambda I - K \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$). Indeed, the equalities (6) hold also in the case where K is Riesz [23]. An analogous result holds if we assume that K is a commuting quasi-nilpotent operator, see [16, Theorem 11], since quasi-nilpotent operators are Riesz. These results may fail if K is not assumed to commute, see [16, Example 12]. Browder's theorem for T and S transfers successfully to the tensor product $T \bigotimes S$ [17, Theorem 6]. In [16] it is also shown that Browder's theorem holds for a Hilbert space operator $T \in L(H)$ if T is reduced by its finite dimensional eigenspaces.

Browder's theorem entails the continuity of some mappings. To see this, we need some preliminary definitions. Let (σ_n) be a sequence of compacts subsets of $\mathbb{C}$ and define canonically its *limit inferior* by

$$\liminf \sigma_n := \{\lambda \in \mathbb{C} : \text{there exists } \lambda_n \in \sigma_n \text{ with } \lambda_n \to \lambda\}.$$

Define the *limit superior* of (σ_n) by

$$\limsup \sigma_n := \{\lambda \in \mathbb{C} : \text{there exists } \lambda_{n_k} \in \sigma_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}.$$

A mapping φ, defined on $L(X)$ whose values are compact subsets of $\mathbb{C}$ is said to be *upper semi-continuous at* T (respectively, *lower semi-continuos* a T) provided that if $T_n \to T$, in the norm topology, then $\limsup \varphi(T_n) \subseteq \varphi(T)$ (respectively, $\varphi(T) \subseteq \liminf \varphi(T_n)$). If the map φ is both upper and lower semi-continuous then φ is said to be *continuos* at T. In this case we write $\lim_{n\in\mathbb{N}} \varphi(T_n) = \varphi(T)$. In the following result we consider mappings that associate to an operator its Browder spectrum or its Weyl spectrum.

Theorem 3.9. *[12] If $T \in L(X)$ then the following assertions hold:*

(i) *The map $T \in L(X) \to \sigma_{\rm b}(T)$ is continuous at T_0 if and only if Browder's theorem holds for T_0.*

(ii) *If Browder's theorem holds for T_0 then the map $T \in L(X) \to \sigma(T)$ is continuous at T_0.*

By contrast, we see now that Browder's theorem is equivalent to the discontinuity of some other mappings. Recall that *reduced minimum modulus* of a non-zero operator T is defined by

$$\gamma(T) := \inf_{x\notin \ker\, T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)}.$$

In the following result we use the concept of gap metric, see [19] for details.

Theorem 3.10. *[3] For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies Browder's theorem;*

(ii) *the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at every $\lambda \in \Delta(T)$ in the gap metric;*

(iii) *the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta(T)$;*

(iv) *the mapping $\lambda \to (\lambda I - T)(X)$ is not continuous at every $\lambda \in \Delta(T)$ in the gap metric.*

4 a-Browder's theorem

An approximation point version of Browder's theorem is given by the so-called a-Browder's theorem. A bounded operator $T \in L(X)$ is said to satisfy *a-Browder's theorem* if

$$\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T),$$

or equivalently, by (1), if

$$\operatorname{acc} \sigma_{\rm a}(T) \subseteq \sigma_{\rm wa}(T).$$

Define

$$p_{00}^a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T) = \{\lambda \in \sigma_{\rm a}(T) : \lambda I - T \in \mathcal{B}_+(X)\},$$

and let us consider the following set:

$$\Delta_a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm wa}(T).$$

Since $\lambda I - T \in W_a(X)$ implies that $(\lambda I - T)(X)$ is closed, we can write

$$\Delta_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W_a(X),\, 0 < \alpha(\lambda I - T)\}.$$

It should be noted that the set $\Delta_a(T)$ may be empty. This is, for instance, the case of a right shift on $\ell^2(\mathbb{N})$. We have

$$p_{00}^a(T) \subseteq \pi_{00}^a(T) \quad \text{for all } T \in L(X),$$

and

$$p_{00}^a(T) \subseteq \Delta_a(T) \subseteq \sigma_{\rm a}(T) \quad \text{for all } T \in L(X).$$

Theorem 4.1. *For a bounded operator $T \in L(X)$, a-Browder's theorem holds for T if and only if $p_{00}^a(T) = \Delta_a(T)$. In particular, a-Browder's theorem holds whenever $\Delta_a(T) = \emptyset$.*

A precise description of operators satisfying a-Browder's theorem may be given in terms of SVEP at certain sets.

Theorem 4.2. *If $T \in L(X)$ the following statements hold:*

(i) *T satisfies a-Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{\rm wa}(T)$.*

(ii) *T^* satisfies a-Browder's theorem if and only if T^* has SVEP at every $\lambda \notin \sigma_{\rm ws}(T)$.*

(iii) *If T has SVEP at every $\lambda \notin \sigma_{\rm ws}(T)$ then a-Browder's theorem holds for T^*.*

(iv) *If T^* has SVEP at every $\lambda \notin \sigma_{\rm wa}(T)$ then a-Browder's theorem holds for T.*

Since $\sigma_{\rm wa}(T) \subseteq \sigma_{\rm w}(T)$, from Theorem 4.2 and Theorem 3.1 we readily obtain:

$$a\text{-Browder's theorem for } T \Rightarrow \text{Browder's theorem for } T,$$

while

$$\text{SVEP for either } T \text{ or } T^* \Rightarrow a\text{-Browder's theorem holds for both } T, T^*. \quad (7)$$

Note that the reverse of the assertions (iii) and (iv) of Theorem 3.1 generally do not hold. An example of unilateral weighted shifts T on $\ell^p(\mathbb{N})$ for which a-Browder's theorem holds for T (respectively, a-Browder's theorem holds for T^*) and such that SVEP fails at some points $\lambda \notin \sigma_{\rm ws}(T)$ (respectively, at some points $\lambda \notin \sigma_{\rm wa}(T)$) may be found in [4].

The implication of (7) may be considerably extended as follows.

Theorem 4.3. *[11], [2] Let $T \in L(X)$ and suppose that T or T^* satisfies SVEP. Then a-Browder's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$, i.e. $\sigma_{\rm wa}(f(T)) = \sigma_{\rm ub}(f(T))$. Furthermore,*

$$\sigma_{\rm ws}(f(T)) = \sigma_{\rm lb}(f(T)), \quad \sigma_{\rm w}(f(T)) = \sigma_{\rm b}(f(T)),$$

and the spectral mapping theorem holds for all the spectra $\sigma_{\rm wa}(T)$, $\sigma_{\rm ws}(T)$ and $\sigma_{\rm w}(T)$.

Theorem 4.3 is an easy consequence of the fact that $f(T)$ satisfies Browder's theorem and that the spectral mapping theorem holds for the Browder spectrum and semi-Browder spectra, see [1, Theorem 3.69 and Theorem 3.70]. In general, the spectral mapping theorems for the Weyl spectra $\sigma_{\rm w}(T)$, $\sigma_{\rm wa}(T)$ and $\sigma_{\rm ws}(T)$ are liable to fail. Moreover, Browder's theorem and the spectral mapping theorem are independent. In [16, Example 6] is given an example of an operator T for which the spectral mapping theorem holds for $\sigma_{\rm w}(T)$ but Browder's theorem fails for T. Another example [16, Example 7] shows that there exist operators for which Browder's theorem holds, while the spectral mapping theorem for the Weyl spectrum fails.

The following results are analogous to the results of Theorem 3.3, and give a precise spectral picture of a-Browder's theorem.

Theorem 4.4. *[4], [10] For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies a-Browder's theorem;*

(ii) $\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\rm a}(T)$;

(iii) $\Delta_a(T) \subseteq \partial\sigma_{\rm a}(T)$, *$\partial\sigma_{\rm a}(T)$ the topological boundary of $\sigma_{\rm a}(T)$;*

(iv) $\sigma_{\rm a}(T) = \sigma_{\rm wa}(T) \cup \sigma_{\rm k}(T)$;

(v) $\Delta_a(T) \subseteq \sigma_{\rm k}(T)$;

(vi) $\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\rm k}(T)$;

(vii) $\Delta_a(T) \subseteq \sigma_{\rm sa}(T)$;

(viii) $\Delta_a(T) \subseteq \operatorname{iso}\sigma_{\rm sa}(T)$.

We also have:

Theorem 4.5. *[3] $T \in L(X)$ satisfies a-Browder's theorem if and only if*

$$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{wa}}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(T). \tag{8}$$

Analogously, a-Browder's theorem holds for T^* if and only if

$$\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ws}}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{s}}(T). \tag{9}$$

The results established above have some nice consequences.

Corollary 4.6. *Suppose that T^* has SVEP. Then $\Delta_a(T) \subseteq \mathrm{iso}\,\sigma(T)$.*

Proof. We can suppose that $\Delta_a(T)$ is non-empty. If T^* has SVEP then a-Browder' s theorem holds for T, so by Theorem 4.4 $\Delta_{\mathrm{a}} \subseteq \mathrm{iso}\,\sigma_{\mathrm{a}}(T)$. Moreover, by Corollary 3.19 of [1] for all $\lambda \in \Delta_{\mathrm{a}}(T)$ we have $\mathrm{ind}(\lambda I - T) \leq 0$, so $0 < \alpha(\lambda I - T) \leq \beta(\lambda I - T)$, and hence $\lambda \in \sigma_{\mathrm{s}}(T)$. Now, if $\lambda \in \Delta_{\mathrm{a}}(T)$ the SVEP for T^* entails by Theorem 2.2 that $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{s}}(T)$, and hence $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{s}}(T) \cap \mathrm{iso}\,\sigma_{\mathrm{a}}(T) = \mathrm{iso}\,\sigma(T)$. ∎

Corollary 4.7. *Suppose that $T \in L(X)$ has SVEP and $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) = \emptyset$. Then*

$$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{k}}(T). \tag{10}$$

Analogously, if T^* has SVEP and $\mathrm{iso}\,\sigma_{\mathrm{s}}(T) = \emptyset$, then

$$\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ws}}(T) = \sigma_{\mathrm{k}}(T). \tag{11}$$

Proof. If T has SVEP then a-Browder's theorem holds for T. Since $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) = \emptyset$, by Theorem 4.4 we have we have $\Delta_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \emptyset$. Therefore $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{wa}}(T)$ and this set coincides with the spectrum $\sigma_{\mathrm{k}}(T)$, see [1, Chapter 2].

If T^* has SVEP and $\mathrm{iso}\,\sigma_{\mathrm{s}}(T) = \emptyset$, then $\mathrm{iso}\,\sigma_{\mathrm{a}}(T^*) = \mathrm{iso}\,\sigma_{\mathrm{s}}(T) = \emptyset$ and the first part implies that $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{wa}}(T^*) = \sigma_{\mathrm{k}}(T^*)$. By duality we then easily obtain that $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ws}}(T) = \sigma_{\mathrm{k}}(T)$. ∎

The first part of the previous corollary applies to a right weighted shift T on $\ell^p(\mathbb{N})$, where $1 \leq p < \infty$. In fact, if the spectral radius $r(T) > 0$ then $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) = \emptyset$, since $\sigma_{\mathrm{a}}(T)$ is a closed annulus (possible degenerate), see Proposition 1.6.15 of [20], so (10) holds, while if $r(T) = 0$ then, trivially, $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{k}}(T) = \{0\}$. Of course, the equality (11) holds for any left weighted shift. Corollary 4.7 also applies to non-invertible isometry, since for these operators we have $\sigma_{\mathrm{a}}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, see [20].

As in Theorem 3.4, some characterizations of operators satisfying a-Browder's theorem may be given in terms of the quasi-nilpotent part $H_0(\lambda I - T)$.

Theorem 4.8. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *a-Browder's theorem holds for T.*

(ii) *$H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta_a(T)$.*

(iii) *$H_0(\lambda I - T)$ is closed for every $\lambda \in \Delta_a(T)$.*

Note that in Theorem 4.8 does not appear a characterization of a-Browder's theorem in terms of the analytic core $K(\lambda I - T)$, analogous to that established in Theorem 3.4. The authors in [4] have proved only the following implication:

Theorem 4.9. *If $K(\lambda I - T)$ is finite-codimensional for all $\lambda \in \Delta_a(T)$ then a-Browder's theorem holds for T.*

It would be of interest to prove whenever the converse of the result of Theorem 4.9 holds.

Define
$$\sigma_2(T) := \sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{k}}(T).$$

Note that
$$\sigma_2(f(T)) \subseteq f(\sigma_2(T)) \quad \text{for all } f \in \mathcal{H}(\sigma(T)),$$

see Lemma 3.5 of [10]. A necessary and sufficient condition for the spectral mapping for $\sigma_2(T)$ is given in the next result.

Theorem 4.10. *[10] The spectral mapping theorem holds for $\sigma_2(T)$ precisely when $ind(\lambda I - T) \cdot ind(\mu I - T) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C}$ such that $\lambda I - T \in \Phi_+(X)$ and $\mu I - T \in \Phi_-(X)$.*

Using the spectral mapping theorem for $\sigma_{\mathrm{a}}(T)$, see Theorem 2.48 of [1], it is easy to derive the following result analogous to that established in Theorem 3.8

Theorem 4.11. *[10] [12] Suppose that both $T \in L(X)$ and $S \in L(X)$ satisfy a-Browder's theorem and $f \in \mathcal{H}(\sigma(T))$. Then we have:*

(i) *a-Browder's theorem holds for $f(T)$ if and only if $f(\sigma_2(T)) = \sigma_2(f(T))$.*

(ii) *a-Browder's theorem holds for the direct sum* $T \oplus S$ *if and only if* $\sigma_2(T) \cup \sigma_2(S) = \sigma_2(T \oplus S)$.

(iii) *a-Browder's theorem holds for the direct sum* $T \oplus S$ *if and only if* $\sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{wa}}(S) = \sigma_{\mathrm{wa}}(T \oplus S)$.

Also a-Browder's theorem survives under perturbation of Riesz operators K commuting with T, where T satisfies a-Browder's theorem. In fact, we have

$$\sigma_{\mathrm{wa}}(T+K) = \sigma_{\mathrm{wa}}(T), \quad \sigma_{\mathrm{ub}}(T+K) = \sigma_{\mathrm{ub}}(T),$$

see [23]. Similar equalities hold for quasi-nilpotent perturbations Q commuting with T, so that a-Browder's theorem holds for $T+Q$.

Note that a-Browder's theorem transfers successfully to $p(T)$, p a polynomial, if we assume that $p(\sigma_{wa}(T)) = \sigma_{wa}(p(T))$. In fact, we have:

Theorem 4.12. *[12] If the map* $T \in L(X) \to \sigma_{\mathrm{wa}}(T)$ *is continuous at* T_0 *then a-Browder's theorem holds for* T_0. *Furthermore, if a-Browder's theorem holds for* T *and* p *is a polynomial then a-Browder's theorem holds for* $p(T)$ *if and only if* $p(\sigma_{wa}(T)) = \sigma_{wa}(p(T))$.

We conclude by noting that, as Browder's theorem, a-Browder's theorem is equivalent to the discontinuity of some mappings.

Theorem 4.13. *[4] For a bounded operator* $T \in L(X)$ *the following statements are equivalent:*

(i) T *satisfies a-Browder's theorem;*

(ii) *the mapping* $\lambda \to \ker(\lambda I - T)$ *is not continuous at every* $\lambda \in \Delta_a(T)$ *in the gap metric;*

(iii) *the mapping* $\lambda \to \gamma(\lambda I - T)$ *is not continuous at every* $\lambda \in \Delta_a(T)$*;*

(iv) *the mapping* $\lambda \to (\lambda I - T)(X)$ *is not continuous at every* $\lambda \in \Delta_a(T)$ *in the gap metric.*

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1.2. SOBRE LA INVERTIBILIDAD DE DRAZIN

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ON DRAZIN INVERTIBILITY

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Abstract. The left Drazin spectrum and the Drazin spectrum coincide with the upper semi-B-Browder spectrum and the B-Browder spectrum, respectively. We also prove that some spectra coincide whenever T or T^* satisfies the single-valued extension property.

1. Introduction and preliminaries

Throughout this note $L(X)$ will denote the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. The operator $T \in L(X)$ is said to be *upper semi-Fredholm* if its kernel ker T is finite-dimensional and the range $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm* if $T(X)$ is finite-codimensional. If either T is upper or lower semi-Fredholm, then T is said to be a *semi-Fredholm operator*, while T is said to be a *Fredholm operator* if it is both upper and lower semi-Fredholm. If $T \in L(X)$ is semi-Fredholm, the classical *index* of T is defined by $\text{ind}\,(T) := \dim \ker T - \operatorname{codim} T(X)$.

The concept of semi-Fredholm operators has been generalized by Berkani ([9], [13] and [11]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi-B-Fredholm*, (resp. *B-Fredholm, upper semi-B-Fredholm*, *lower semi-B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([13]). This enables one to define the index of a semi-B-Fredholm operator as $\text{ind}\, T = \text{ind}\, T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be a *Weyl operator* if T is a Fredholm operator having index 0. A bounded operator $T \in L(X)$ is said to be *B-Weyl* if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl. The *Weyl spectrum* and the *B-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}$$

and

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not } B\text{-Weyl}\}.$$

Recall that the *ascent* of an operator $T \in L(X)$ is defined as the smallest nonnegative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such an integer does not

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exist, we put $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest nonnegative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$; see [1, Theorem 3.3]. Moreover, if $\lambda \in \mathbb{C}$, the condition $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ is equivalent to saying that λ is a pole of the resolvent. In this case λ is an eigenvalue of T and an isolated point of the spectrum $\sigma(T)$; see [17, Prop. 50.2].

The concept of Drazin invertibility [14] has been introduced in a more abstract setting than operator theory [14]. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) precisely when $p(T) = q(T) < \infty$ and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent; see [19, Corollary 2.2] and [18, Prop. A]. Every B-Fredholm operator T admits the representation $T = T_0 \oplus T_1$, where T_0 is Fredholm and T_1 is nilpotent [11], so every Drazin invertible operator is B-Fredholm.

The concept of Drazin invertibility for bounded operators may be extended as follows.

Definition 1.1. *$T \in L(X)$ is said to be left Drazin invertible if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed; while $T \in L(X)$ is said to be right Drazin invertible if $q := q(T) < \infty$ and $T^q(X)$ is closed.*

It should be noted that the condition $q = q(T) < \infty$ does not entails that $T^q(X)$ is closed; see Example 5 of [21]. Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$, then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$; see [17, Prop. 50.2].

The *left Drazin spectrum* is then defined as

$$\sigma_{\mathrm{ld}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

the *right Drazin spectrum* is defined as

$$\sigma_{\mathrm{rd}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\},$$

and the *Drazin spectrum* is defined as

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

Obviously, $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{ld}}(T) \cup \sigma_{\mathrm{rd}}(T)$.

A bounded operator $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is Fredholm and $p(T) = q(T) < \infty$ (resp. T is upper semi-Fredholm and $p(T) < \infty$, T is lower semi-Fredholm and $q(T) < \infty$). Every Browder operator is Weyl and hence, if

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}$$

denotes the Browder spectrum of T, then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. In the sequel by $\sigma_{\mathrm{usb}}(T)$ we shall denote the *upper semi-Browder spectrum* of T defined by

$$\sigma_{\mathrm{usb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\}.$$

Clearly, every bounded below operator $T \in L(X)$ (T injective with closed range) is upper semi-Browder, while every surjective operator is lower semi-Browder. The classical *approximate point spectrum* of T will be denoted by $\sigma_{\mathrm{a}}(T)$ while by $\sigma_{\mathrm{s}}(T)$ we shall denote the *surjectivity spectrum* of T.

It is natural to extend the concept of semi-Browder operators as follows: A bounded operator $T \in L(X)$ is said to be *B-Browder* (resp. *upper semi-B-Browder*, *lower semi-B-Browder*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Browder (resp. upper semi-Browder, lower semi-Browder). The respective B-Browder spectra are denoted by $\sigma_{\mathrm{bb}}(T)$, $\sigma_{\mathrm{usbb}}(T)$ and $\sigma_{\mathrm{lsbb}}(T)$.

The main result of this paper establishes that $T \in L(X)$ is B-Browder (respectively, upper semi-B-Browder, lower semi-Browder) if and only if T is Drazin invertible (respectively, left Drazin invertible, right Drazin invertible); consequently $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{d}}(T)$, $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ld}}(T)$ and $\sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{rd}}(T)$. We also prove that many of the spectra before introduced coincide whenever T, or its dual T^*, satisfies the single-valued extension property.

2. SVEP and semi-B-Browder spectra

A useful tool in the Fredholm theory is given by the localized single-valued extension property. This property has an important role in local spectral theory; see the recent monographs by Laursen and Neumann [20] and Aiena [1].

Definition 2.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ of λ_0, the only analytic function $f : U \to X$ that satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that the localized SVEP is inherited by the restriction to closed invariant subspaces; i.e., if T has SVEP at λ_0 and M is a closed T-invariant subspace of X, then $T|M$ has SVEP at λ_0. Moreover, the set $\Sigma(T)$ of all points $\lambda \in \mathbb{C}$ such that T does not have SVEP at λ is an open set contained in the interior of the spectrum of T. Consequently, if T has SVEP at each point λ of an open punctured disc $\mathbb{D} \setminus \{\lambda_0\}$ centered at λ_0, then T also has SVEP at λ_0.

We have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually,

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda; \tag{2}$$

see [1, Theorem 3.8]. Furthermore, from the definition of localized SVEP it is easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually,

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{4}$$

Remark 2.2. The implications (1), (2), (3) and (4) are actually equivalences if T is a semi-Fredholm operator; see [5] or [1, Chapter 3].

Lemma 2.3. *If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:*

(i) *there exists $n \geq p+1$ such that $T^n(X)$ is closed;*
(ii) *$T^n(X)$ is closed for all $n \geq p$.*

Proof. Define $c_i'(T) := \dim(\ker T^i / \ker T^{i+1})$. Clearly, $p = p(T) < \infty$ entails that $c_i'(T) = 0$ for all $i \geq p$, so $k_i(T) := c_i'(T) - c_{i+1}'(T) = 0$ for all $i \geq p$. The equivalence then easily follows from [21, Lemma 12]. □

Define

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.4. $T \in L(X)$ is said to be *quasi-Fredholm of degree d* if there exists $d \in \mathbb{N}$ such that:

(a) $\mathrm{dis}(T) = d$,
(b) $T^n(X)$ is a closed subspace of X for each $n \geq d$,
(c) $T(X) + \ker T^d$ is a closed subspace of X.

It should be noted that by Proposition 2.5 of [13] every semi-B-Fredholm operator is quasi-Fredholm.

Theorem 2.5. *For every $T \in L(X)$ the following statements are equivalent:*

(i) *T is left Drazin invertible;*
(ii) *There exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below;*
(iii) *T is semi-B-Fredholm and T has SVEP at 0.*

Dually, if $T \in L(X)$ the following statements are equivalent:

(iv) *T is right Drazin invertible;*
(v) *there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is onto;*
(vi) *T is semi-B-Fredholm and T^* has SVEP at 0.*

Proof. (i) $\Leftrightarrow$ (ii) Suppose that T is left Drazin invertible. Then $p = p(T) < \infty$ and $T^{p+1}(X)$ is closed. From Lemma 2.3 it follows that $T^p(X)$ is closed. By [1, Lemma 3.2] we have $\ker T \cap T^p(X) = \ker T_{[p]} = \{0\}$, so $T_{[p]}$ is injective. The range of $T_{[p]}$ is closed, since it coincides with $T^{p+1}(X)$; hence $T_{[p]}$ is bounded below, so the condition (ii) is satisfied.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is bounded below. Let us consider an element $x \in \ker T^{n+1}$. Clearly, $T(T^n x) = 0$ so $T^n x \in \ker T$. Since $T^n x \in T^n(X)$ it then follows that $T^n x \in \ker T \cap T^n(X) = \ker T_{[n]} = \{0\}$; thus $x \in \ker T^n$. Therefore, $\ker T^{n+1} = \ker T^n$, so T has finite ascent $p := p(T) \leq n$. The range of $T_{[n]}$ is the closed subspace $T^{n+1}(X)$, with $p+1 \leq n+1$. Therefore $T^{p+1}(X)$ is closed; thus T is left Drazin invertible.

(ii) $\Leftrightarrow$ (iii) Assume (i) or equivalently (ii). Then T has SVEP at 0, since $p(T) < \infty$ and $T_{[n]}$ is upper semi-Fredholm, so T is upper semi-B-Fredholm.

Conversely, suppose that T is semi-B-Fredholm and T has SVEP at 0. By Proposition 3.2 of [10] if T quasi-Fredholm, in particular if T is semi-B-Fredholm, then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is semi-regular (i.e., it has closed range and its kernel is contained in the range of each iterate of $T_{[n]}$). Since the restriction $T_{[n]}$ has SVEP at 0, from Theorem 2.49 of [1] it then follows that $T_{[n]}$ is bounded below.

(iv) $\Leftrightarrow$ (v) If $q := q(T) < \infty$, then $T(T^q(X)) = T^{q+1}(X) = T^q(X)$, so $T_{[q]}$ is onto. Moreover, $T^q(X)$ is closed by assumption. Conversely, if (v) holds, then $T^{n+1}(X) = T^n(X)$ so $q := q(T) \leq n$. Obviously, $T^q(X) = T^n(X)$ is closed.

(v) $\Leftrightarrow$ (vi). Assume (v), or equivalently (iv). Since $q := q(T) < \infty$, then T^* has SVEP at 0 and, clearly, $T_{[n]}$ is lower semi-Fredholm, so (vi) holds. The opposite implication has been proved in [2, Theorem 2.7]. $\square$

Corollary 2.6. *$T \in L(X)$ is Drazin invertible if and only if T is semi-B-Fredholm and both T and T^* have SVEP at* 0.

The condition that T, or T^*, has SVEP at 0 for semi-B-Fredholm operators, more generally for quasi-Fredholm operators, may be characterized as follows:

Theorem 2.7. [2] *Suppose that $T \in L(X)$ is quasi-Fredholm. Then the following statements are equivalent:*

(i) *T has SVEP at* 0*;*

(ii) *$\sigma_{\mathrm{a}}(T)$ does not cluster at* 0*.*

Dually, if $T \in L(X)$ is quasi-Fredholm, then the following statements are equivalent:

(iii) *T^* has SVEP at* 0*;*

(iv) *$\sigma_{\mathrm{s}}(T)$ does not cluster at* 0*.*

Given $n \in \mathbb{N}$ let us denote by $T_n : X/\ker T^n \to X/\ker T^n$ the quotient map defined canonically by $T_n\,\hat{x} := Tx$ for each $\hat{x} \in X := X/\ker T^n$, where $x \in \hat{x}$.

Lemma 2.8. *Suppose that $T \in L(X)$ and $T^n(X)$ is closed for some $n \in \mathbb{N}$. If $T_{[n]}$ is upper semi-Fredholm, then T_n is upper semi-Fredholm and ind $T_n =$ ind $T_{[n]}$. Analogous statements hold if $T_{[n]}$ is assumed to be lower semi-Fredholm, Weyl, upper or lower semi-Browder, respectively.*

Proof. The operator $[T^n] : X/\ker T^n \to T^n(X)$ defined by

$$[T^n]\hat{x} = T^n x, \quad \text{where } x \in \hat{x},$$

is a bijection, and it easy to check that $[T^n]T_n = T_{[n]}[T^n]$, from which the statements follow. $\square$

Theorem 2.9. *Suppose that $T \in L(X)$. Then the following equivalences hold:*

(i) *T is upper semi-B-Browder if and only if T is left Drazin invertible.*

(ii) *T is lower semi-B-Browder if and only if T is right Drazin invertible.*

(iii) *T is B-Browder if and only if T is Drazin invertible.*

Proof. (i) Trivially, every bounded below operator is upper semi-Browder. By Theorem 2.5 if T is left Drazin invertible, then T is upper semi-B-Browder.

Conversely, suppose that T is upper semi-B-Browder. By Lemma 2.8, then T_n is upper semi-Browder for some $n \in \mathbb{N}$ and hence by Remark 2.2 the condition $p(T_n) < \infty$ is equivalent to saying that $\sigma_{\mathrm{a}}(T_n)$ does not cluster at 0. Let $\mathbb{D}(0, \varepsilon)$ be an open ball centered at 0 such that $\mathbb{D}(0, \varepsilon) \setminus \{0\}$ does not contain points of $\sigma_{\mathrm{a}}(T_n)$, so

$$\ker(\lambda I - T_n) = \{0\} \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{5}$$

Since the restriction $T|\ker T^n$ is nilpotent we also have that $\mathbb{D}(0, \varepsilon) \setminus \{0\} \subseteq \rho(T|\ker T^n)$, $\rho(T|\ker T^n)$ the resolvent of $T|\ker T^n$, so

$$(\lambda I - T)(\ker T^n) = \ker T^n \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{6}$$

Since for all $0 < |\lambda| < \varepsilon$ we also have $\ker(\lambda I - T|\ker T^n) = \{0\}$, it then easily follows that $\ker(\lambda I - T) = \{0\}$, i.e. $\lambda I - T$ is injective for all $0 < |\lambda| < \varepsilon$.

We show now that $(\lambda I - T)(X)$ is closed for all $0 < |\lambda| < \varepsilon$.

Set $\hat{X} := X/\ker T^n$ and let $w \in (\lambda I - T)(X)$ be arbitrary. Then there exists $x \in X$ such that $w = (\lambda I - T)x$ and hence $\hat{w} = (\lambda I - T_n)\hat{x} \in (\lambda I - T_n)(\hat{X})$. Since $\lambda \notin \sigma_{\mathrm{a}}(T_n)$, then $(\lambda I - T_n)(\hat{X})$ is closed, and hence there exists a sequence $(w_n) \subset X$ such that $(\lambda I - T_n)\hat{w_n} \to \hat{w}$ as $n \to +\infty$; thus

$$(\lambda I - T)w_n - w \to z_n \in \ker T^n.$$

From (6) we know that there exists $y_n \in \ker T^n$ such that $z_n = (\lambda I - T)y_n$, and hence

$$(\lambda I - T)w_n - (\lambda I - T)y_n = (\lambda I - T)(w_n - y_n) \to w,$$

so that $(\lambda I - T)(X)$ is closed. We have shown that $\lambda I - T$ is bounded below for all $0 < |\lambda| < \varepsilon$ and, consequently, 0 is an isolated point of $\sigma_{\mathrm{a}}(T)$. This implies that T has SVEP at 0 and since by assumption T is upper semi-B-Browder from Theorem 2.5, we then conclude that T is left Drazin invertible.

(ii) By Theorem 2.5, if T is right Drazin invertible, then there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is onto and hence lower semi-Browder.

Conversely, suppose that T is lower semi-B-Browder and let $n \in \mathbb{N}$ such that $T_{[n]}$ is lower semi-Browder. By Lemma 2.8, then T_n is lower semi-Browder and hence the condition $q(T_n) < \infty$ is equivalent to saying that $\sigma_{\mathrm{s}}(T_n)$ does not cluster at 0. Let $\mathbb{D}(0,\varepsilon)$ be an open ball centered at 0 such that $\mathbb{D}(0,\varepsilon) \setminus \{0\}$ does not contain points of $\sigma_{\mathrm{s}}(T_n)$. As in the proof of part (i) we have $(\lambda I - T)(\ker T^n) = \ker T^n$ for all $0 < |\lambda| < \varepsilon$. We show that $(\lambda I - T)(X) = X$ for all $0 < |\lambda| < \varepsilon$. Since $\lambda I - T_n$ is onto, for each $x \in X$ there exists $y \in X$ such that $(\lambda I - T_n)\hat{y} = \hat{x}$ and hence

$$x - (\lambda I - T)y \in \ker T^n = (\lambda I - T)(\ker T^n).$$

Consequently, there exists $z \in \ker T^n$ such that $x - (\lambda I - T)y = (\lambda I - T)z$, from which it follows that

$$x = (\lambda I - T)(z + y) \in (\lambda I - T)(X).$$

We have proved that $\lambda I - T$ is onto for all $0 < |\lambda| < \varepsilon$; thus $\sigma_{\mathrm{s}}(T)$ does not cluster at 0 and consequently T^* has SVEP at 0. By Theorem 2.5 we then conclude that T is right Drazin invertible.

(iii) Clear. □

Corollary 2.10. *For every $T \in L(X)$ we have*

$$\sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{ld}}(T), \quad \sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{rd}}(T), \quad \sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{d}}(T).$$

3. Browder type theorems

Let us denote by $USBF^-(X)$ the class of all upper semi-B-Fredholm operators with index less than or equal to 0, while by $LSBF^+(X)$ we denote the class of all lower semi-B-Fredholm operators with index greater than or equal to 0. Set

$$\sigma_{\mathrm{usbf}^-}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin USBF^-(X)\}$$

and

$$\sigma_{\mathrm{lsbf}^+}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin LSBF^+(X)\}.$$

Theorem 3.1. *If $T \in L(X)$, then the following equalities hold:*
(i) $\sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{usbf}^-}(T) \cup acc\,\sigma_{\mathrm{a}}(T)$.
(ii) $\sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{lsbf}^+}(T) \cup acc\,\sigma_{\mathrm{s}}(T)$.
(iii) $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bw}}(T) \cup acc\,\sigma(T)$.

Proof. The proof of the equalities (i), (iii) may be found in [6] and [7]. To show the equality (ii), we observe first that

$$\sigma_{\mathrm{lsbf}^+}(T) \subseteq \sigma_{\mathrm{rd}}(T). \tag{7}$$

Indeed, if $\lambda \notin \sigma_{\mathrm{rd}}(T)$, then, by Theorem 2.5, $\lambda I - T_{[n]}$ is onto some $n \in \mathbb{N}$, hence lower semi-Fredholm and

$$\mathrm{ind}(\lambda I - T) = \mathrm{ind}(\lambda I - T_{[n]}) = \alpha(\lambda I - T_{[n]}) \geq 0;$$

thus $\lambda \notin \sigma_{\mathrm{lsbf}^+}(T)$.

By Corollary 2.10, in order to show the inclusion $\sigma_{\mathrm{lsbb}}(T) \supseteq \sigma_{\mathrm{lsbf}^+}(T) \cup \mathrm{acc}\,\sigma_{\mathrm{s}}(T)$ we need only to prove that $\mathrm{acc}\,\sigma_{\mathrm{s}}(T) \subseteq \sigma_{\mathrm{lsbb}}(T)$. If $\lambda \notin \sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{rd}}(T)$, then $\lambda I - T$ is right Drazin invertible, and hence by Theorem 2.5, $\lambda I - T$ is T is semi-B-Fredholm with $q(\lambda I - T) < \infty$. By Corollary 4.8 of [16] it then follows that $\lambda I - T$ is onto in a punctured disc centered at λ; thus $\lambda \notin \mathrm{acc}\,\sigma_{\mathrm{s}}(T)$.

To show the opposite inclusion $\sigma_{\mathrm{lsbb}}(T) \subseteq \sigma_{\mathrm{lsbf}^+}(T) \cup \mathrm{acc}\,\sigma_{\mathrm{s}}(T)$, suppose that $\lambda \notin \sigma_{\mathrm{lsbf}^+}(T) \cup \mathrm{acc}\,\sigma_{\mathrm{s}}(T)$. Since $\lambda \notin \mathrm{acc}\,\sigma_{\mathrm{s}}(T)$, then T^* has SVEP at λ. Since $\lambda I - T$ is lower semi-B-Fredholm by Theorem 2.5, then $\lambda I - T$ is right Drazin invertible. By Corollary 2.10, then $\lambda \notin \sigma_{\mathrm{rd}}(T) = \sigma_{\mathrm{lsbb}}(T)$, so the equality (ii) is proved. □

A bounded operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. Denote by $\sigma_{\mathrm{usf}^-}(T)$ the *essential approximate point spectrum* of T, defined as the complement in $\mathbb{C}$ of the set of all λ such that $\lambda I - T$ is upper semi-Fredholm with $\mathrm{ind}\,T \leq 0$. The operator $T \in L(X)$ is said to satisfy *a-Browder's theorem* if $\sigma_{\mathrm{usf}^-}(T) = \sigma_{\mathrm{ub}}(T)$; see for instance [4].

According to [12], a bounded operator $T \in L(X)$ is said to satisfy the *generalized Browder's theorem* if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$, while $T \in L(X)$ is said to satisfy the *generalized a-Browder's theorem* if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{ld}}(T)$.

Note that in all the papers concerning generalized Browder's theorems (see for instance [7], [15], [12], [8]), there is no trace of the role of B-Browder spectra. Our Corollary 2.10 shows that this is only apparent. In fact, by Corollary 2.10 we have:

$$\text{generalized Browder's theorem holds for } T \Leftrightarrow \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T),$$

while

$$\text{generalized } a\text{-Browder's theorem holds for } T \Leftrightarrow \sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{usbb}}(T).$$

Browder's theorem may be characterized by localized SVEP: Browder's theorem (resp. generalized Browder's theorem) holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$ ([3]) (resp. T has SVEP at every $\lambda \notin \sigma_{\mathrm{bw}}(T)$, see [7]), while a-Browder's theorem (resp. generalized a-Browder's theorem) holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{usf}^-}(T)$ ([4]) (resp. T has SVEP at every $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$, see([6]). The inclusions $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$ and $\sigma_{\mathrm{usf}^-}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T)$ immediately entail that the generalized Browder's theorem implies Browder's theorem, and, analogously, the generalized a-Browder's theorem implies a-Browder's theorem. The main result of a very recent paper [8] proves that Browder's theorem and the generalized Browder's theorem (respectively, a-Browder's theorem and the generalized a-Browder's theorem) are equivalent. These results may be shown in a few lines as follows:

Theorem 3.2. *For every $T \in L(X)$ the following equivalences hold:*
(i) $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) \Leftrightarrow \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T)$.
(ii) $\sigma_{\mathrm{usf}^-}(T) = \sigma_{\mathrm{ub}}(T) \Leftrightarrow \sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{usbb}}(T)$.

Proof. (i) We have only to show the implication $\Rightarrow$. Assume that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. Clearly, $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{bb}}(T)$ for all $T \in L(X)$. To show the opposite inclusion, assume that $\lambda_0 \notin \sigma_{\mathrm{bw}}(T)$, i.e. that $\lambda_0 I - T$ is B-Weyl. By [13, Corollary 3.2], then there exists an open disc $\mathbb{D}$ such that $\lambda I - T$ is Weyl and hence Browder for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. Since $p(\lambda I - T) = q(\lambda I - T) < \infty$, then both T and T^* have SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$, and hence both T and T^* have SVEP at λ_0. By Theorem 2.5, then $\lambda_0 I - T$ is Drazin invertible, or equivalently $\lambda_0 \notin \sigma_{\mathrm{bb}}(T)$. Hence $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T)$.

(ii) Also here it suffices to prove the implication $\Rightarrow$. Assume that $\sigma_{\mathrm{usf}^-}(T) = \sigma_{\mathrm{ub}}(T)$. Clearly, $\sigma_{\mathrm{usbf}^-}(T) \subseteq \sigma_{\mathrm{usf}}(T)$ for all $T \in L(X)$. Suppose that $\lambda_0 \notin \sigma_{\mathrm{usbf}^-}(T)$. Then $\lambda_0 I - T \in USBF^-(X)$ and by [13, Corollary 3.2] there exists an open disc $\mathbb{D}$ such that $\lambda I - T$ is upper semi-Fredholm with index less than or equal to 0 for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. From assumption then $\lambda I - T$ is upper semi-Browder; hence $p(\lambda I - T) < \infty$. Thus, T has SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$ and hence T also has SVEP at λ_0. By Theorem 2.5 we then conclude that $\lambda_0 \notin \sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{usbb}}(T)$, so the equality $\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{usbb}}(T)$ is proved. □

The following result shows that many of the spectra considered before coincide whenever T or T^* has SVEP.

Theorem 3.3. *Suppose that $T \in L(X)$. Then the following statements hold:*
(i) *If T has SVEP, then*

$$\sigma_{\mathrm{lsbf}^+}(T) = \sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bw}}(T). \tag{8}$$

(ii) *If T^* has SVEP, then*

$$\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{9}$$

(iii) *If both T and T^* have SVEP, then*

$$\sigma_{\mathrm{usbf}^-}(T) = \sigma_{\mathrm{lsbf}^+}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T). \tag{10}$$

Proof. (i) By Theorem 3.1 and Corollary 2.10 we have

$$\sigma_{\mathrm{lsbf}^+}(T) \subseteq \sigma_{\mathrm{lsbb}}(T) = \sigma_{\mathrm{rd}}(T) \subseteq \sigma_{\mathrm{d}}(T).$$

We show now that $\sigma_{\mathrm{d}}(T) \subseteq \sigma_{\mathrm{lsbf}^+}(T)$. Assume that $\lambda \notin \sigma_{\mathrm{lsbf}^+}(T)$. We may assume $\lambda = 0$. Since T is lower semi-B-Fredholm and since T^* has SVEP, in particular T^* has SVEP at 0, by Theorem 2.5 then T is right Drazin invertible or, equivalently, lower semi-B-Browder. Therefore there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is lower semi-Fredholm and $q(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\operatorname{ind} T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{\mathrm{lsbf}^+}(T)$, we also have $\operatorname{ind} T_{[n]} \geq 0$ from which we obtain $\operatorname{ind} T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $p(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence T is B-Browder. By part (iii) of Theorem 2.9 then T is Drazin invertible, so $0 \notin \sigma_{\mathrm{d}}(T)$, as desired. Finally, since T has SVEP by which the T satisfies the generalized Browder's theorem, we have $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$ and the equalities (8) are proved.

(ii) The inclusion $\sigma_{\mathrm{lsbf}^-}(T) \subseteq \sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{ld}}(T) \subseteq \sigma_{\mathrm{d}}(T)$ holds for every $T \in L(X)$ by Theorem 3.1 and Corollary 2.10.

We show that $\sigma_{\mathrm{d}}(T) \subseteq \sigma_{\mathrm{usbf}^-}(T)$. Suppose that $\lambda \notin \sigma_{\mathrm{usbf}^-}(T)$ and assume that $\lambda = 0$. Since T is upper semi-B-Fredholm, then there exists $n \in \mathbb{N}$ such that $T_{[n]}$ is upper semi-Fredholm. The restriction $T_{[n]} := T|T^n(X)$ has SVEP, in particular has SVEP at 0 and hence, see Remark 2.2, $p(T_{[n]}) < \infty$. By Theorem 3.4 of [1] it then follows that $\operatorname{ind} T_{[n]} \leq 0$. On the other hand, since $\lambda \notin \sigma_{\mathrm{lsbf}^+}(T)$, we also have $\operatorname{ind} T_{[n]} \geq 0$ from which we obtain $\operatorname{ind} T_{[n]} = 0$. This implies, again by Theorem 3.4 of [1], that also $q(T_{[n]}) < \infty$, so that $T_{[n]}$ is Browder and hence T is B-Browder. By part (iii) of Theorem 2.9, then T is Drazin invertible, so $0 \notin \sigma_{\mathrm{d}}(T)$, as desired. Finally, since T has SVEP, then T satisfies the generalized Browder's theorem, so $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$.

(iii) Clear from parts (i), (ii). □

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1.3. ESPECTROS B-BROWDER Y LA SVEP LOCAL

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B-Browder spectra and localized SVEP

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Abstract. In this paper we study the relationships between the B-Browder spectra and some other spectra originating from Fredholm theory and B-Fredholm theory. This study is done by using the localized single valued extension property. In particular, we shall see that many spectra coincide in the case that a bounded operator T, or its dual T^*, or both, admits the single valued extension property.

Keywords Localized SVEP, B-Fredholm theory, semi B-Browder spectra

Mathematics Subject Classification (2000) 47A10 · 47A11 · (Secondary) 47A53 · 47A55

1 Introduction and terminology

An important class of operators in Fredholm theory is the class of (upper, lower) semi-Browder operators defined on Banach spaces ([17], [18]). This

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class of operators has been studied in [5] by using methods of local spectral theory, in particular these operators have been characterized by means of a localized version of the so-called single-valued extension property (SVEP). In this paper we extend these results to the class of semi B-Browder operators, defined according the B-Fredholm theory introduced by Berkani and coauthors ([9], [10], [11]). The characterizations of semi B-Browder operators in terms of localized SVEP are then used for obtaining many relationships between some spectra originating from Fredholm theory and B-Fredholm theory. In particular, we show that if an operator T, or its dual T^*, satisfies SVEP then many of these spectra coincide. We lso consider the special case where the boundary of the spectrum coincide with the approximate point spectrum, or with the surjectivity spectrum.

Throughout this paper $L(X)$ will denote the algebra of all bounded linear operators acting on an infinite- dimensional complex Banach space X. For $T \in L(X)$ we denote by $N(T)$ the null space of T, by $\alpha(T) = \dim N(T)$ the nullity of T, by R(T)=T(X) the range of T and by $\beta(T) = \operatorname{codim} R(T) = \dim X/R(T)$ the defect of T. The *ascent* $p = p(T)$ of an operator T is defined as the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$. If such an integer does not exist, we put $p(T) = \infty$. Analogously, the *descent* $q = q(T)$ is defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$, and if such an integer does not exist, we put $q(T) = \infty$. An operator $T \in L(X)$ is said to be *Fredholm* (resp. *upper semi -Fredholm, lower semi-Fredholm*), if $\alpha(T)$, $\beta(T)$ are both finite (resp. $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). T is said to be *semi-Fredholm* if T is either an upper semi-Fredholm or a lower semi-Fredholm operator. Other two important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follow, $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder, lower semi-Browder*) if T is a Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm) and both $p(T)$, $q(T)$ are finite (resp. $p(T) < \infty$, $q(T) < \infty$).

Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. According Berkani ([10] and [11]), T is said to be semi *B-Fredholm* (resp. *B-Fredholm*, *upper semi B-Fredholm*, *lower semi B-Fredholm*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, viewed as a operator from the space $R(T^n)$ in to itself, is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm) . Analogously, $T \in L(X)$ is said to be *B-Browder* (resp., *upper semi B-Browder*, *lower semi B-Browder*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is a Browder operator (resp., upper semi-Browder, lower semi -Browder). Define

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap N(T) \subseteq T^m(X) \cap N(T)\}.$$

The *degree of stable iteration* is defined as $\operatorname{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 1 $T \in L(X)$ is said to be *quasi-Fredholm of degree d*, if there exists $d \in \mathbb{N}$ such that:

(a) $\mathrm{dis}(T) = d$,

(b) $T^n(X)$ is a closed subspace of X for each $n \geq d$,

(c) $T(X) + N(T^d)$ is a closed subspace of X.

For further informations on quasi-Fredholm operators we refer to [10] and [11].

2 Ascent and descent of restrictions

This first section leads with some preliminary results concerning the ascent and the descent of restrictions of T to the ranges of its power. We start first with the following useful lemma ([1, Lemma 3.2]):

Lemma 1 *Let T be a linear operator on a vector space X. Then $p := p(T) \leq m < \infty$ if and only if $N(T^n) \cap T^m(X) = \{0\}$ for all $n \in \mathbb{N}$..*

Suppose now that $T \in L(X)$ and put $T_n := T|T^n(X)$ for all $n \in \mathbb{N}$. Then

$$N(T_{n+1}) = N(T) \cap T^{n+1}(X) \subseteq N(T) \cap T^n(X) = N(T_n) \text{ for all } n \in \mathbb{N}, \quad (1)$$

and

$$R(T_n^m) = R(T^{m+n}) = R(T_m^n) \text{ for all } m,n \in \mathbb{N}. \quad (2)$$

Lemma 2 *Let T be a linear operator on a vector space X. Then the following statements are equivalent:*

(i) *$p(T) < \infty$;*

(ii) *there exists $k \in \mathbb{N}$ such that T_k is injective;*

(iii) *there exists $k \in \mathbb{N}$ such that $p(T_k) < \infty$.*

Proof. (i) $\Leftrightarrow$ (ii) If $p := p(T) < \infty$, by Lemma 1, then $N(T_p) = N(T) \cap T^p(X) = \{0\}$. Conversely, suppose that $N(T_k) = \{0\}$, for some $k \in \mathbb{N}$. If $x \in N(T^{k+1})$ then $T(T^k x) = 0$, so

$$T^k x \in N(T) \cap T^k(X) = N(T_k) = \{0\}.$$

Hence $x \in N(T^k)$. This shows that $N(T^{k+1}) \subseteq N(T^k)$. The opposite inclusion is true for every operator, thus $N(T^{k+1}) = N(T^k)$ and consequently $p(T) \leq k$.

(ii) $\Leftrightarrow$ (iii) The implication (ii) $\Rightarrow$ (iii) is obvious. To show the opposite implication, suppose that $\nu := p(T_k) < \infty$. By Lemma 1 and by using the equality (2) we have:

$$\begin{aligned}\{0\} &= N(T_k) \cap R({T_k}^\nu) = (N(T) \cap R(T^k)) \cap R({T_k}^\nu) = N(T) \cap R({T_k}^\nu) \\ &= N(T) \cap R(T^{\nu+k}) = N(T_{\nu+k}),\end{aligned}$$

so that the equivalence $\Leftrightarrow$ (iii) is proved. $\square$

A dual result holds for the descent:

Lemma 3 *Let T be a linear operator on a vector space X. Then the following statements are equivalent:*

(i) $q(T) < \infty$;

(ii) *there exists $k \in \mathbb{N}$ such that T_k is onto;*

(iii) *there exists $k \in \mathbb{N}$ such that $q(T_k) < \infty$.*

Proof. (i) $\Leftrightarrow$ (ii) Suppose that $q := q(T) < \infty$. Then

$$T^q(X) = T^{q+1}(X) = T(T^q(X)) = R(T_q),$$

hence T_q is onto. Conversely, if T_k is onto for some $k \in \mathbb{N}$ then

$$T^{k+1}(X) = T(T^k(X)) = R(T_k) = T^k(X),$$

thus $q(T) \leq k$.

The implication (ii) $\Rightarrow$ (iii) is obvious. We show (iii) $\Rightarrow$ (i). Suppose that $\nu := q(T_k) < \infty$ for some $k \in \mathbb{N}$. Then ${T_k}^{\nu}(X) = {T_k}^{\nu+1}(X)$, i.e. $T^{k+\nu}(X) = T^{\nu+k+1}(X)$, hence $q(T) \leq k + \nu$. $\square$

Remark 1 As observed in the proof of Lemma 2 if $p := p(T) < \infty$ then $N(T_p) = \{0\}$ and from the inclusion (1) it is obvious that $N(T_j) = \{0\}$ for all $j \geq p$. Conversely, if $N(T_k) = \{0\}$ for some $k \in \mathbb{N}$ then $p(T) < \infty$ and $p(T) \leq k$. Hence, if $p(T) < \infty$ then

$$p(T) = \inf\{k \in \mathbb{N} : T_k \text{ is injective}\}.$$

Analogously, if $q := q(T) < \infty$ then T_j is onto for all $j \geq q$. Conversely, if T_k is onto for some $k \in \mathbb{N}$ then $q(T) \leq k$, so that

$$q(T) = \inf\{k \in \mathbb{N} : T_k \text{ is onto}\}.$$

Definition 2 $T \in L(X)$, X a Banach space, is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be *right Drazin invertible* if $q := q(T) < \infty$ and $T^q(X)$ is closed.

It should be noted that the condition $q = q(T) < \infty$ does not entails that $T^q(X)$ is closed, see Example 5 of [15]. Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection P_0 associated with the spectral set $\{0\}$, see [13, Prop. 50.2]. Later we shall see that left Drazin invertible operator, as well as every right Drazin invertible operator, is semi B-Fredholm.

Lemma 4 *If $T \in L(X)$ and $p = p(T) < \infty$ then the following statements are equivalent:*

(i) *There exists a natural $n \geq p+1$ such that $T^n(X)$ is closed;*

(ii) *$T^n(X)$ is closed for all $n \geq p$.*

Proof. Define $c_i'(T) := \dim(N(T^i)/N(T^{i+1})$. Clearly, $p = p(T) < \infty$ entails that $c_i'(T) = 0$ for all $i \geq p$, so $k_i(T) := c_i'(T) - c_{i+1}'(T) = 0$ for all $i \geq p$. The equivalence then easily follows from [15, Lemma 12]. □

Recall that a bounded operator $T \in L(X)$ on a Banach space is called *bounded below* if T is injective and has closed range. The concept of left (respectively, right) Drazin invertibility may be view as the topological counterpart of the property of having finite ascent (respectively, finite descent). In fact we have:

Theorem 1 *If $T \in L(X)$ then we have:*

(i) *T is left Drazin invertible if and only if there exists a $k \in \mathbb{N}$ such that $T^k(X)$ is closed and T_k is bounded below. In this case $T^j(X)$ is closed and T_j is bounded below for all naturals $j \geq k$.*

(ii) *T is right Drazin invertible if and only if there exists a $k \in \mathbb{N}$ such that $T^k(X)$ is closed and T_k is onto. In this case $T^j(X)$ is closed and T_j is onto for all naturals $j \geq k$.*

(iii) *T is Drazin invertible if and only if there exists a $k \in \mathbb{N}$ such that $T^k(X)$ is closed and T_k is invertible. In this case $T^j(X)$ is closed and T_j is invertible for all naturals $j \geq k$.*

Proof. (i) Suppose $p := p(T) < \infty$ and that $T^{p+1}(X)$ closed. Then T_p is injective and $R(T_p) = T^{p+1}(X)$ is closed. Conversely, if T_k is bounded below for some $k \in \mathbb{N}$ then, by Lemma 2, $p := p(T) < \infty$ and by Remark 1 we have $p \leq k$, and hence $p+1 \leq k+1$. Since $R(T_k) = T^{k+1}(X)$ is closed then, by Lemma 4, $T^{p+1}(X)$ is closed and consequently T is left Drazin invertible. The last assertion is clear, by Remark 1, T_j is injective for all $j \geq k$ and $T^j(X)$ is closed, again by Lemma 4.

(ii) Suppose that $q := q(T) < \infty$ and $T^q(X)$ is closed then $R(T_q) = T^{q+1}(X) = T^q(X)$, so T_q is onto. Conversely, suppose that $T^k(X)$ is closed and T_k is onto for some $k \in \mathbb{N}$. Then, by Lemma 2, $q = q(T) < \infty$ and $q+1 \leq k+1$. By Lemma 4 then $T^q(X)$ is closed and hence T is right Drazin invertible. By Lemma 4 then $T^j(X)$ is closed for all $j \geq k$, and by Remark 1 T_j is onto for all $j \geq k$.

(iii) Clear. □

Also here, if T is left Drazin invertible then

$$p(T) = \inf\{k \in \mathbb{N} : T^k(X) \text{ is closed and } T_k \text{ is bounded below}\},$$

while, if T is right Drazin invertible then

$$q(T) = \inf\{k \in \mathbb{N} : T^k(X) \text{ is closed and } T_k \text{ is onto}\}.$$

Observe that also the property of being quasi-Fredholm may be described in terms of restrictions. Recall that $T \in L(X)$ is said to be *semi-regular* if $T(X)$ is closed and $N(T) \subseteq T^n(X)$ for all $n \in \mathbb{N}$.

Theorem 2 *$T \in L(X)$ is quasi-Fredholm if and only if the exists $k \in \mathbb{N}$ such that $T^k(X)$ is closed and T_k is semi-regular. In this case T_j is semi-regular for all $j \geq k$.*

Proof. The equivalence is due to Berkani [10, Proposition 3.2]. To prove the last statement, suppose that T_k is semi-regular for some $k \in \mathbb{N}$. For all $j \geq k$ then

$$N(T_j) \subseteq N(T_k) \subseteq R(T_k^n) = T^{k+n+1}(X) \quad \text{for all } n \in \mathbb{N}.$$

In particular,

$$N(T_j) \subseteq T^{k+(j-k)+n+1}(X) = T^{j+n+1}(X) = R(T_j^n),$$

for all $n \in \mathbb{N}$. Moreover, since T_k^n is semi-regular for all $n \in \mathbb{N}$, see Corollary [1, Corollary 1.17], it then follows that $R(T_j)$ is closed for all $j \geq k$. Hence T_j is semi-regular. □

Clearly, every semi-regular operator is quasi-Fredholm. It should be noted that both Theorem 1 and Theorem 2 provide a very clear picture of the relationship between the concepts of quasi-Fredholm operators and Drazin (left, right) invertibility: every bounded below operator, as well as every surjective operator, is semi-regular, so from Theorem 2 and Theorem 1 we easily deduce that every left Drazin invertible operator, as well as every right Drazin invertible operator is quasi-Fredholm. Actually, every semi B-Fredholm operator is quasi-Fredholm, see Proposition 2.5 of [11].

3 SVEP

We now define a basic property, introduced by Finch [12], and later studied extensively by Aiena and coauthors ([1],[3], [4], [5] and [7]). A bounded operator $T \in L(X)$ on a complex Banach space X is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at λ_0), if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T) f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that the localized SVEP is inherited by the restriction to closed invariant subspaces, i.e. if T has SVEP at λ_0 and M is a closed T-invariant subspace of X then $T|M$ has SVEP at λ_0. Moreover, if $\mathscr{H}(\sigma(T))$ denotes the set of all complex-valued functions which are locally analytic on an open set containing $\sigma(T)$, for every $f \in \mathscr{H}(\sigma(T))$ then $f(T)$ has the SVEP, see [1, Theorem 2.40]. We have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{4}$$

see [1, Theorem 3.8].

Remark 2 The implications (3), (4), are actually equivalences if T is a quasi-Fredholm operator, see [3].

The class of B-Browder operators may be described in terms of SVEP:

Theorem 3 *Let $T \in L(X)$. Then the following properties are equivalent:*

(i) $\lambda_0 I - T$ *is left Drazin invertible;*

(ii) $\lambda_0 I - T$ *is upper semi B-Browder;*

(iii) $\lambda_0 I - T$ *is quasi-Fredholm operator having finite ascent;*

(iv) $\lambda_0 I - T$ *is quasi-Fredholm and T has the SVEP at λ_0.*

Proof. Clearly, T has the SVEP at λ_0 if and only if $\lambda_0 I - T$ has the SVEP at 0, so in the proof we can suppose $\lambda_0 = 0$.

(i) $\Rightarrow$ (ii) Clearly, if T is left Drazin invertible then, by Theorem 1, there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed, T_n is bounded below, and hence upper semi-Browder.

(ii) $\Rightarrow$ (iii) As already observed, T is quasi-Fredholm. Moreover, since T_n is upper semi-Browder for some $n \in \mathbb{N}$ then $p(T_n) < \infty$, and this entails, by Lemma 2, that $p(T) < \infty$.

(iii) $\Rightarrow$ (i) If T quasi-Fredholm and $p := p(T) < \infty$ then, by Remark 1, T_n is injective for all $n \geq p$. Moreover, if d is the degree of T then $T^n(X)$ is closed for all $n \geq d$, so T_n is bounded below for n sufficiently large. By Theorem 1 then T is left Drazin invertible.

(iii) $\Leftrightarrow$ (iv) This is clear by Remark 2. □

The equivalence (i) $\Leftrightarrow$ (ii) of Theorem 3 has been proved in [4] (see also [10]) . Our proof is much more simple.

Theorem 4 *Let $T \in L(X)$ Then the following properties are equivalent:*

(i) $\lambda_0 I - T$ *is right Drazin invertible;*

(ii) $\lambda_0 I - T$ *is lower semi B-Browder;*

(iii) $\lambda_0 I - T$ *is quasi-Fredholm having finite descent;*

(iv) $\lambda_0 I - T$ *is quasi-Fredholm and T^* has SVEP at λ_0.*

Proof. Also here we can assume that $\lambda_0 = 0$.

(i) $\Rightarrow$ (ii) Clearly, if T is right Drazin invertible then, by Theorem 1, there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed, T_n is onto, and hence a lower semi-Browder operator,

(ii) $\Rightarrow$ (iii) Clearly, T is quasi-Fredholm. Moreover, since T_n is lower semi-Browder for some $n \in \mathbb{N}$ then $q(T_n) < \infty$, and by Lemma 3 this is equivalent to saying that $q(T) < \infty$.

(iii) $\Rightarrow$ (i) Suppose that T is quasi-Fredholm and $q := q(T) < \infty$. As observed in Remark 1 then T_n is onto for all $n \geq q$. As $T^n(X)$ is closed for all $n \geq d$, where d is the degree of T. By Theorem 1 it then follows that T is right Drazin invertible.

(iii) $\Leftrightarrow$ (iv) This follows from Remark 2. $\square$

Also the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 4 has been observed in [10] and proved by using different methods in [4]. The proof given here is much more simple. It should be noted that for Hilbert spaces operators instead of considering the dual T^* of T is more appropriate to consider the Hilbert adjoint T'. Since, as observed in [2], T^* has SVEP at λ if and only if T' has SVEP at λ , in the case of Hilbert space operators the dual T^* of T may be replaced by T'.

Corollary 1 *Let $T \in L(X)$. Then the following properties are equivalent:*

(i) $\lambda_0 I - T$ *is Drazin invertible ;*

(ii) $\lambda_0 I - T$ *is B-Browder;*

(iii) $\lambda_0 I - T$ *is quasi-Fredholm and both T, T^* have the SVEP at λ_0.*

4 Some Relationships between spectra

The classes of operators defined in the previous section motivate the definitions of several spectra. The *upper semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

The *lower semi-Browder spectrum* is defined by

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Browder}\}.$$

From the classical Fredholm theory we have $\sigma_{\mathrm{ub}}(T)=\sigma_{\mathrm{lb}}(T^*)$ and $\sigma_{\mathrm{lb}}(T)=\sigma_{\mathrm{ub}}(T^*)$. The B-Fredholm spectrum is by

$$\sigma_{\mathrm{bf}}(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is not B-Fredholm}\},$$

the *upper B-Browder spectrum* spectrum of $T\in L(X)$ is defined by

$$\sigma_{\mathrm{ubb}}(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is not upper B-Browder}\},$$

the *lower B-Browder spectrum* spectrum is defined by

$$\sigma_{\mathrm{lbb}}(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is not lower B-Browder}\},$$

while the *B-Browder spectrum* is defined, by

$$\sigma_{\mathrm{bb}}(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is not B-Browder}\}.$$

Clearly, $\sigma_{\mathrm{bb}}(T)=\sigma_{\mathrm{ubb}}(T)\cup\sigma_{\mathrm{lbb}}(T)$. An obvious consequence of Corollary 1 is that $\sigma_{\mathrm{bb}}(T)$ coincides with the *Drazin spectrum* $\sigma_{\mathrm{d}}(T)$ of T. Moreover, by Theorem 3, $\sigma_{\mathrm{ubb}}(T)$ coincides with $\sigma_{\mathrm{ld}}(T)$, the *left Drazin spectrum* of T, and by Theorem 4 $\sigma_{\mathrm{lbb}}(T)=\sigma_{\mathrm{d}}(T)$, the *right Drazin spectrum* of T. The *quasi-Fredholm spectrum* of $T\in L(X)$ is defined by

$$\sigma_{\mathrm{qf}}(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is not quasi-Fredholm }\}.$$

Hence,

$$\sigma_{\mathrm{qf}}(T)\subseteq\sigma_{\mathrm{bf}}(T)\subseteq\sigma_{\mathrm{bb}}(T). \tag{5}$$

Note that all the spectra in (5) may be empty. This is the case where the spectrum of T is a finite set of poles of the resolvent (i.e. T is algebraic, see [1, Theorem 3.83]. In this case, $\sigma_{\mathrm{bb}}(T)=\sigma_{\mathrm{d}}(T)$ is obviously empty. Furthermore, all the spectra in (5) are compact subsets of $\mathbb{C}$, see [10, Corollary 3.8]. In the sequel by ∂K we denote the boundary of $K\subset\mathbb{C}$.

Theorem 5 *If $T\in L(X)$ we have*

(i) $\partial\sigma_{\mathrm{bb}}(T)\subseteq\sigma_{\mathrm{qf}}(T)$,

(ii *If If $\sigma_{\mathrm{bf}}(T)$ is connected then $\sigma_{\mathrm{bb}}(T)$ is connected,*

Proof. (i) Obviously, we can assume that $\sigma_{\mathrm{bb}}(T)$ is not empty. Suppose that $\lambda\in\partial\sigma_{\mathrm{bb}}(T)$. We claim that both T and T^* have SVEP at λ. Let $f:\mathbb{D}_\lambda\to X$ a analytic function on the open disc $\mathbb{D}_\lambda$ centered at λ, such that

$$(\mu I-T)f(\mu)=0 \quad \text{for each } \mu\in\mathbb{D}_\lambda.$$

Let $\rho_{\mathrm{bb}}(T):=\mathbb{C}\setminus\sigma_{\mathrm{bb}}(T)$. Then $\mathbb{D}_\lambda\cap\rho_{\mathrm{bb}}(T)\neq\emptyset$ and $\mu I-T$ is a B-Browder operator for all $\mu\in\mathbb{D}_\lambda\cap\rho_{\mathrm{bb}}(T)$, so by Corollary 1, both T and T^* have SVEP at every $\mu\in\mathbb{D}_\lambda\cap\rho_{\mathrm{bb}}(T)$. On the other hand, $\mathbb{D}_\lambda\cap\rho_{\mathrm{bb}}(T)$ is open and for each $\mu \in \mathbb{D}_\lambda \cap \rho_{bb}(T)$ there exists an open disc $\mathbb{D}_\mu \subseteq \mathbb{D}_\lambda \cap \rho_{bb}(T)$ centered at μ such that $f : \mathbb{D}_\mu \to X$ is analytic and the equation $(\eta I - T) f(\eta) = 0$ holds for all $\eta \in \mathbb{D}_\mu$. Since T has the SVEP at μ, it then follows that $f \equiv 0$ in $\mathbb{D}_\mu$. This implies, via the identity theorem for analytic functions, that $f \equiv 0$ in $\mathbb{D}_\lambda$. Thus, T has the SVEP at λ. In similar way, T^* has the SVEP at λ. Now, if $\lambda \notin \sigma_{qf}(T)$, then $\lambda I - T$ is quasi-Fredholm and the SVEP for T and T^* at λ implies that $\lambda \notin \sigma_{bb}(T)$, a contradiction.

(ii) Assume that $\sigma_{bf}(T)$ is connected and $\sigma_{bb}(T)$ is not connected. Suppose, for instance $\sigma_{bb}(T) = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are two closed non-empty set such that $\Omega_1 \cap \Omega_2 = \emptyset$. Since $\sigma_{bf}(T) \subseteq \sigma_{bb}(T)$ and $\sigma_{bf}(T)$ is connected then $\sigma_{bf}(T)$ is contained either in Ω_1 or Ω_2. Suppose $\sigma_{bf}(T) \subseteq \Omega_1$. The set $\rho_{bf}(T)$ is open and hence may be decomposed in maximal open connected components. Evidently, Ω_2 is contained in the unbounded component Ω of $\rho_{bf}(T) = \mathbb{C} \setminus \sigma_{bf}(T)$ which intersects the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. By Theorem 3.3 and Theorem 3.4 of [8] both T and T^* have SVEP either at every point or at no point of a component of $\rho_{bf}(T)$. Since T and T^* have SVEP at the points of the resolvent, it then follows that T and T^* have SVEP at all points of Ω. In particular, T and T^* have SVEP at every $\lambda \in \Omega_2$. But $\lambda \notin \sigma_{bf}(T)$, hence $\lambda I - T$ is quasi-Fredholm. By Corollary 1 it then follows that $\lambda I - T$ is B-Browder, a contradiction, since $\Omega_2 \subseteq \sigma_{bb}(T)$. □

A bounded operator $T \in L(X)$ is said to be a *Weyl operator* if T is a Fredholm operator having index 0; $T \in L(X)$ is said to be *upper semi-Weyl* if T upper semi-Fredholm with index $\operatorname{ind} T \leq 0$; T is said to be *lower semi-Weyl* if T is lower semi-Fredholm with $\operatorname{ind} T \geq 0$. The *Weyl spectrum* and is defined, by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\},$$

the *upper semi-Weyl spectrum* and *lower semi-Weyl spectrum* are defined, respectively, by

$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\},$$

and

$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Weyl}\}.$$

It is known from the classical Fredholm theory that $\sigma_{uw}(T) = \sigma_{lw}(T^*)$ and $\sigma_{lw}(T) = \sigma_{uw}(T^*)$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*), if for some integer $n \geq 0$ the range $T^n(X)$ is closed and T_n is Weyl (respectively, upper semi-Weyl, lower semi-Weyl).

The *B-Weyl spectrum* is defined by

$$\sigma_{bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\};$$

the *upper semi B-Weyl spectrum* and the *lower semi B-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{ubw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\},$$

and

$$\sigma_{\mathrm{lbw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\}.$$

Clearly, $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{ubw}}(T) \cup \sigma_{\mathrm{lbw}}(T)$. For an operator $T \in L(X)$, we set

$$\Xi(T) := \{\lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda\}.$$

Trivially, $\Xi(T)$ is empty whenever T has the SVEP. Moreover, the set $\Xi(T)$ is an open set contained in interior of the spectrum of T.

Theorem 6 *Let $T \in L(X)$. Then we have:*

$$\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{qf}}(T) \cup \Xi(T) = \sigma_{\mathrm{ubw}}(T) \cup \Xi(T) \tag{6}$$

and

$$\sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{qf}}(T) \cup \Xi(T^*) = \sigma_{\mathrm{lbw}}(T) \cup \Xi(T^*) \tag{7}$$

Moreover,

$$\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{bw}}(T) \cup \Xi(T) = \sigma_{\mathrm{bw}}(T) \cup \Xi(T^*). \tag{8}$$

Proof. We show the first equality in (6). Suppose that $\lambda \in \sigma_{\mathrm{qf}}(T) \cup \Xi(T)$. Then $\lambda \in \sigma_{\mathrm{qf}}(T)$ or T does not have SVEP at λ. In the first case $\lambda \in \sigma_{\mathrm{ubb}}(T)$, since $\sigma_{\mathrm{qf}}(T) \subseteq \sigma_{\mathrm{ubb}}(T)$. Also the second case entails that $\lambda \in \sigma_{\mathrm{ubb}}(T)$, otherwise by Lemma 2 we would have $p(\lambda I - T) < \infty$ and hence T has SVEP at λ. Therefore, $\sigma_{\mathrm{qf}}(T) \cup \Xi(T) \subseteq \sigma_{\mathrm{ubb}}(T)$. Conversely, if $\lambda \notin \sigma_{\mathrm{qf}}(T) \cup \Xi(T)$ then, by Theorem 3, $\lambda I - T$ is upper semi-Browder, so $\lambda \notin \sigma_{\mathrm{ubb}}(T)$.
To show the second equality in (6), observe first that $\sigma_{\mathrm{ubb}}(T) \subseteq \sigma_{\mathrm{ubw}}(T) \cup \Xi(T)$, since $\sigma_{\mathrm{qf}}(T) \subseteq \sigma_{\mathrm{ubw}}(T)$. Let $\lambda \notin \sigma_{\mathrm{ubb}}(T)$. Then $\lambda I - T$ is upper semi B-Browder, in particular upper semi B-Weyl, so $\lambda \notin \sigma_{\mathrm{ubw}}(T)$. Clearly, T has SVEP, since by Lemma 2 $p(\lambda I - T) < \infty$, hence $\lambda \notin \Xi(T)$. Hence both equalities in (6) are proved.
The equalities in (7) can be proved by means of similar arguments, just use Lemma 3. Analogously, the equalities in (8) may be proved by using both Lemma 2 and Lemma 3. □

Corollary 2 *Let $T \in L(X)$. Then we have:*

(i) *If T has the SVEP then*

$$\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ubb}}(T), \tag{9}$$

and

$$\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{lbw}}(T). \tag{10}$$

(ii) *If T^* has the SVEP then*

$$\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{lbb}}(T), \tag{11}$$

and

$$\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ubw}}(T). \tag{12}$$

(iii) *If both T, T^* have SVEP then*

$$\begin{aligned}\sigma_{\mathrm{qf}}(T) &= \sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{bb}}(T)\\ &= \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{ubw}}(T).\end{aligned}$$

Proof. (i) The equalities in (9) are clear from Theorem 6. Also the first equality in (10) is clear from Theorem 6. We show the equality $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{lbb}}(T)$. Clearly, $\sigma_{\mathrm{lbb}}(T) \subseteq \sigma_{\mathrm{bb}}(T)$. Conversely, if $\lambda \notin \sigma_{\mathrm{lbb}}(T)$ then $q(\lambda I - T) < \infty$, by Lemma 3. But $\lambda I - T$ is quasi-Fredholm and the SVEP at λ implies by Theorem 3 that $\lambda I - T$ is upper semi B-Browder, hence $p(\lambda I - T) < \infty$, so that $\lambda \notin \sigma_{\mathrm{bb}}(T)$. Therefore, $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{lbb}}(T)$. To show the equality $\sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{lbw}}(T)$ we need only to prove that $\sigma_{\mathrm{lbb}}(T) \subseteq \sigma_{\mathrm{lbw}}(T)$. If $\lambda \notin \sigma_{\mathrm{lbw}}(T)$ then $\lambda I - T$ is semi B-Fredholm, hence quasi-Fredholm. By Theorem 3 the SVEP of T at λ implies that $\lambda I - T$ is semi B-Browder, hence $\lambda \notin \sigma_{\mathrm{lbb}}(T)$.

(ii) The equalities (11) and (12) can be proved in a similar way of part (i).

(iii) The equalities are consequence of part (i) and part (ii). □

Corollary 2 improves the results of Theorem 3.3 of [4].

Remark 3 Since the SVEP for T (respectively, for T^*) implies that $f(T)$ (respectively, $f(T^*) = f(T)^*$) has SVEP for all $f \in \mathscr{H}(\sigma(T))$, then the equalities established in Corollary 2 holds for $f(T)$.

A bounded operator $T \in L(X)$ is said to satisfy *generalized a-Browder's theorem* if the equality $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$ $(= \sigma_{\mathrm{ubb}}(T))$ holds. Note that generalized a-Browder's theorem implies that the equalities $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bb}}(T))$ hold, namely T satisfies *generalized Browder's theorem*, see for instance [6]. Generalized *a*-Browder's theorem for T is equivalent to so-called *a-Browder's theorem* for T, which means that $\sigma_{\mathrm{ub}}(T)$ coincides with $\sigma_{\mathrm{uw}}(T))$ (see for a simple proof [4]). This implies the so-called *Browder's theorem* for T, namely the equality $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$ holds for T.

The *approximate point spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

the *surjectivity spectrum* of T is defined by

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\},$$

the *Kato spectrum* is defined by

$$\sigma_{\mathrm{k}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

By Theorem 2 we have

$$\sigma_{\mathrm{qf}}(T) \subseteq \sigma_{\mathrm{k}}(T) \subseteq \sigma_{\mathrm{a}}(T) \cap \sigma_{\mathrm{s}}(T). \tag{13}$$

In the sequel, the set of all accumulation points of $K \subseteq \mathbb{C}$ will be denoted by $\operatorname{acc} K$. From the definition of localized SVEP it is easily seen that $\Xi(T) \subseteq \operatorname{acc} \sigma_{\mathrm{a}}(T)$, and dually $\Xi(T^*) \subseteq \operatorname{acc} \sigma_{\mathrm{s}}(T)$.

Corollary 3 *If $T \in L(X)$ then we have*

(i) $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{qf}}(T) \cup acc\ \sigma_{\mathrm{a}}(T)$.

(ii) $\sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{qf}}(T) \cup acc\ \sigma_{\mathrm{s}}(T)$.

(iii) $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{qf}}(T) \cup acc\ \sigma(T)$.

Proof. (i) Clearly, by Theorem 6, $\sigma_{\mathrm{ubb}}(T) \subseteq \sigma_{\mathrm{qf}}(T) \cup \operatorname{acc} \sigma_{\mathrm{a}}(T)$. To show the opposite inclusion let $\lambda \notin \sigma_{\mathrm{qf}}(T) \cup \operatorname{acc}\ \sigma_{\mathrm{a}}(T)$. Then $\lambda I - T$ is quasi-Fredholm and T has SVEP at λ, so $\lambda I - T$ is upper semi B-Browder, by Theorem 3, i.e. $\lambda \notin \sigma_{\mathrm{ubb}}(T)$.

(ii) By Theorem 6, $\sigma_{\mathrm{lbb}}(T) \subseteq \sigma_{\mathrm{qf}}(T) \cup \operatorname{acc} \sigma_{\mathrm{s}}(T)$. Conversely, if $\lambda \notin \sigma_{\mathrm{qf}}(T) \cup \operatorname{acc}\ \sigma_{\mathrm{s}}(T)$. Then $\lambda I - T$ is quasi-Fredholm and T^* has SVEP at λ, so $\lambda I - T$ is lower semi B-Browder, by Theorem 4, i.e. $\lambda \notin \sigma_{\mathrm{lbb}}(T)$.

(iii) This follows from part (i), part (ii), and from the equalities $\sigma(T) = \sigma_{\mathrm{a}}(T) \cup \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{ubb}}(T) \cup \sigma_{\mathrm{lbb}}(T)$. □

Note that if $\operatorname{acc}\ \sigma_{\mathrm{a}}(T) = \emptyset$ then $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{qf}}(T)$. The next two results shows that this equality also holds whenever $\sigma_{\mathrm{a}}(T)$ coincides with the boundary the spectrum $\partial\sigma(T)$ (generally, $\sigma_{\mathrm{a}}(T)$ contains $\partial\sigma(T)$, see [1, Theorem 2.42]).

Theorem 7 *Let $T \in L(X)$ be an operator for which $\sigma_a(T) = \partial\sigma(T) \subseteq acc\ \sigma(T)$. Then*

$$\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{k}}(T). \tag{14}$$

Proof. The assumption entails that T has SVEP. Indeed, T has the SVEP at every point of the boundary as well as at every point λ which belongs to the remaining part of the spectrum, since $\lambda \notin \sigma_a(T)$. Therefore the equalities $\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{ubw}}(T)$ hold by Corollary 2.

We prove that $\sigma_{\mathrm{ubb}}(T) = \sigma_{\mathrm{a}}(T)$. The inclusion $\sigma_{\mathrm{ubb}}(T) \subseteq \sigma_{\mathrm{a}}(T)$ is true for every operator, since by Theorem 1 and Theorem 3 a bounded below operator is upper semi B-Browder. Conversely, suppose that $\lambda \notin \sigma_{\mathrm{ubb}}(T)$. Then $\lambda I - T$ is quasi-Fredholm and the SVEP for T entails, by Theorem 2.7 of [3] that $\sigma_{\mathrm{a}}(T)$ does not cluster at λ. Clearly, $\lambda \notin \sigma_{\mathrm{a}}(T)$, otherwise λ would be an

isolated point of $\sigma_{\mathrm{a}}(T) = \partial\sigma(T)$, contradicting our assumption that every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Hence $\sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{ubb}}(T)$, from which we conclude that $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{ubb}}(T)$.
Evidently, $\sigma_{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ holds for every operator. Now, let $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Then $\lambda \notin \sigma_{\mathrm{ubb}}(T)$ and the SVEP oT at λ, again by Theorem 2.7 of [3], implies that $\lambda \notin \mathrm{acc}\,\sigma_{\mathrm{a}}(T)$. Hence, by Corollary 3, $\lambda \notin \sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{a}}(T)$. Therefore, $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{ub}}(T)$. Finally, the SVEP of T implies, by Remark 3, that $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T)$ and from the inclusions (13) we obtain $\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{k}}(T) = \sigma_{\mathrm{a}}(T)$, so the proof of the equalities (14) is complete. □

Dually we have:

Theorem 8 *Let $T \in L(X)$ be an operator for which $\sigma_s(T) = \partial\sigma(T) \subseteq acc\,\sigma(T)$. Then*

$$\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{k}}(T). \quad (15)$$

Proof. The assumption entails that T^* has SVEP. Indeed, T has the SVEP at every point of the boundary $\partial\sigma(T^*) = \partial\sigma(T)$, as well as at every point λ which belongs to the remaining part of $\sigma(T^*)$, since $\lambda \notin \sigma_s(T)$. Therefore the equalities $\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{lbw}}(T)$ hold by Corollary 2.

We show $\sigma_{\mathrm{lbb}}(T) = \sigma_{\mathrm{s}}(T)$. The inclusion $\sigma_{\mathrm{lbb}}(T) \subseteq \sigma_{\mathrm{s}}(T)$ is clear for every operator, since by Theorem 1 and Theorem 3 a surjective operator is lower semi B-Browder. Conversely, suppose that $\lambda \notin \sigma_{\mathrm{ubb}}(T)$. Then $\lambda I - T$ is quasi-Fredholm and the SVEP for T^* implies, by Theorem 2.11 of [3], that $\sigma_{\mathrm{s}}(T)$ does not cluster at λ. Clearly, $\lambda \notin \sigma_{\mathrm{s}}(T)$, otherwise λ would be an isolated point of $\sigma_{\mathrm{s}}(T) = \partial\sigma(T)$, contradicting the assumption that $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Hence $\sigma_{\mathrm{s}}(T) \subseteq \sigma_{\mathrm{lbb}}(T)$, and consequently $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{lbb}}(T)$. The inclusion $\sigma_{\mathrm{s}}(T) \subseteq \sigma_{\mathrm{lb}}(T)$ holds for every operator. Let $\lambda \notin \sigma_{\mathrm{lb}}(T)$. Then $\lambda \notin \sigma_{\mathrm{lbb}}(T)$ and the SVEP of T^* at λ implies, always by Theorem 2.11 of [3], that $\lambda \notin \mathrm{acc}\,\sigma_{\mathrm{s}}(T)$. Hence, by Corollary 3, $\lambda \notin \sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{s}}(T)$, so the equality $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{lb}}(T)$ is proved. Finally, the SVEP of T^* implies, by Remark 3, that

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*) = \sigma_{\mathrm{ub}}(T^*) = \sigma_{\mathrm{lw}}(T)$$

and from the inclusions (13) we obtain

$$\sigma_{\mathrm{qf}}(T) = \sigma_{\mathrm{k}}(T) = \sigma_{\mathrm{s}}(T),$$

so the proof of the equalities (15) is complete. □

Theorem 7 and Theorem 8 provide an useful tool for determining the various spectra above considered in the case where $T \in L(X)$ is a non-invertible isometry. Indeed, a non-invertible isometry T has SVEP, since its spectrum is the unit disc $\mathbb{D}$ and $\sigma_a(T)$ coincides with the boundary of $\mathbb{D}$, see [14, p. 80].

Theorem 7 also applies to the *Cesáro operator* C_p on the classical Hardy space $H_p(\mathbb{D})$, where $\mathbb{D}$ is the open unit disc and $1 < p < \infty$, defined by

$$(C_p f)(\lambda) := \frac{1}{\lambda}\int_0^\lambda \frac{f(\mu)}{1-\mu}\mathrm{d}\mu \text{ ,for all } f \in H_p(\mathbb{D}) \text{ and } \lambda \in \mathbb{D}.$$

The spectrum of the operator C_p is the closed disc Γ_p centered at $\frac{p}{2}$ with radius $\frac{p}{2}$, see [16], and $\sigma_{\mathrm{a}}(C_p) = \partial\Gamma_p$.

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1.4. TEOREMAS DE WEYL GENERALIZADOS PARA OPERADORES POLAROIDES

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ABSTRACT. In this paper we establish necessary and sufficient conditions on bounded linear operators for which generalized Weyl's theorem, or generalized a-Weyl theorem, holds. We also consider generalized Weyl's theorems in the framework of polaroid operators and obtain improvements of some results recently established in [20] and [29].

1. INTRODUCTION AND TERMINOLOGY

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite- dimensional complex Banach space X. For $T \in L(X)$, we denote by $N(T)$ the null space of T and by $R(T) = T(X)$ the range of T. We denote by $\alpha(T) := \dim N(T)$ the nullity of T and by $\beta(T) := \operatorname{codim} R(T) = \dim X/R(T)$ the defect of T. Other two classical quantities in operator theory are the *ascent* $p = p(T)$ of an operator T, defined as the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$ (if such an integer does not exist, we put $p(T) = \infty$), and the *descent* $q = q(T)$, defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$). An operator $T \in L(X)$ is said to be *Fredholm* (respectively, *upper semi -Fredholm*, *lower semi-Fredholm*), if $\alpha(T)$, $\beta(T)$ are both finite (respectively, $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). $T \in L(X)$ is said to be *semi-Fredholm* if T is either an upper semi-Fredholm or a lower semi-Fredholm operator. If T is semi-Fredholm the *index* of T defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$. Other two important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follows, $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is a Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm) and both $p(T)$, $q(T)$ are finite (respectively, $p(T) < \infty$, $q(T) < \infty$). A bounded operator $T \in L(X)$ is said to be *upper semi-Weyl* (respectively, *lower semi-Weyl*) if T is upper Fredholm operator (respectively, lower semi-Fredholm) and index $\operatorname{ind} T \leq 0$ (respectively, $\operatorname{ind} T \geq 0$). $T \in L(X)$ is said to be *Weyl* if T is both upper and lower semi-Weyl, i.e. T is a Fredholm operator having index 0. The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by $\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}$ and $\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}$.

Since every Browder operator is Weyl then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Analogously, The *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined, respectively, by $\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\}$, and $\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\}$.

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Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. According [16] and [14], T is said to be semi *B-Fredholm* (respectively, *B-Fredholm*, *upper semi B-Fredholm*, *lower semi B-Fredholm*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, viewed as a operator from the space $R(T^n)$ into itself, is a semi-Fredholm operator (respectively, Fredholm, upper semi-Fredholm, lower semi-Fredholm) . Analogously, $T \in L(X)$ is said to be *B-Browder* (respectively, *upper semi B-Browder*, *lower semi B-Browder*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is a Browder operator (respectively, upper semi-Browder, lower semi -Browder). If T_n is a semi-Fredholm operator, it follows from ([14, Proposition 2.1]) that also T_m is semi-Fredholm for every $m \geq n$, and $\operatorname{ind} T_m = \operatorname{ind} T_n$. This enables us to define the *index* of semi B-Fredholm operator T as the index of the semi-Fredholm operator T_n. Thus, a bounded operator $T \in L(X)$ is said to be a *B-Weyl operator* if T is a B-Fredholm operator having index 0, $T \in L(X)$ is said to be *upper semi B-Weyl* if T is upper semi B-Fredholm with index $\operatorname{ind} T \leq 0$, and T is said to be *lower semi B-Weyl* if T is lower semi B-Fredholm with $\operatorname{ind} T \geq 0$. Note that if T is B-Fredholm then also T^* is B-Fredholm with $\operatorname{ind} T^* = -\operatorname{ind} T$.

The classes of operators defined above motivate the definitions of several spectra. The *upper semi B-Browder spectrum* is defined by $\sigma_{\mathrm{ubb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder}\}$. The *lower semi B-Browder spectrum* is defined by $\sigma_{\mathrm{lbb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder}\}$, while the *B-Browder spectrum* is defined, by $\sigma_{\mathrm{bb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\}$. Clearly, $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{ubb}}(T) \cup \sigma_{\mathrm{lbb}}(T)$. The *B-Weyl spectrum* is defined, by $\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}$, the *upper semi B-Weyl spectrum* and *lower semi B-Weyl spectrum* are defined, respectively, by $\sigma_{\mathrm{ubw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\}$, and $\sigma_{\mathrm{lbw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\}$. Two other classes of operators related with semi B-Fredholm operators are the quasi-Fredholm operators and Drazin invertible operators defined in the sequel. $T \in L(X)$ is said to be *Drazin invertible* if $p(T) = q(T) < \infty$. A bounded operator $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be *right Drazin invertible* if $q := q(T) < \infty$ and $T^q(X)$ is closed. Clearly, T is Drazin invertible if and only if T is both right and left Drazin invertible. Define

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T\}.$$

The *degree of stable iteration* is defined as $\operatorname{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 1.1. *$T \in L(X)$ is said to be* quasi-Fredholm of degree d, *if there exists $d \in \mathbb{N}$ such that:*
(a) *$dis(T) = d$,*
(b) *$T^n(X)$ is a closed subspace of X for each $n \geq d$,*
(c) *$T(X) + \ker T^d$ is a closed subspace of X.*

It should be noted that by Proposition 2.5 of [14] every semi B-Fredholm operator is quasi-Fredholm. The *quasi-Fredholm spectrum* is defined as $\sigma_{\mathrm{qf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not quasi-Fredholm}\}$, while the *Drazin spectrum* and the *left Drazin* *spectrum* are defined, respectively, by $\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T$ is not Drazin invertible $\}$, and $\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T$ is not left Drazin invertible$\}$.

Theorem 1.2. *([4]) If $T \in L(X)$ then $\sigma_{\rm ld}(T) = \sigma_{\rm ubb}(T)$ and $\sigma_{\rm d}(T) = \sigma_{\rm bb}(T)$.*

Lemma 1.3. *([26]) If $T \in L(X)$ and $p = p(T) < \infty$ then the following statements are equivalent:*

(i) *There exists $n \geq p+1$ such that $T^n(X)$ is closed;*

(ii) *$T^n(X)$ is closed for all $n \geq p$.*

We now introduce an important property in local spectral theory, see [23] and Chapter 3 of [1]. A bounded operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at λ_0), if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0 \quad$ for all $\lambda \in \mathbb{D}_{\lambda_0}$, is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{1.2}$$

Two important subspaces in local spectral theory are the *analytic core* and the *quasi-nilpotent part* of T. The analytic core $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$, see [1] for information on $K(T)$. The quasi-nilpotent part is defined by $H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$. Note that $N(T^n) \subseteq H_0(T)$ for all $n \in \mathbb{N}$ and (see Chapter 2 of [1]),

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{1.3}$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\rm ap}(T)$ the classical *approximate point spectrum* defined by $\sigma_{\rm ap}(T)$ $:= \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below$\}$. Note that if $\sigma_{\rm s}(T)$ denotes the *surjectivity spectrum* $\sigma_{\rm s}(T) := \{\lambda \in \mathbb{C} : \lambda I - T$ is not onto$\}$, then $\sigma_{\rm ap}(T) = \sigma_{\rm s}(T^*)$ and $\sigma_{\rm s}(T) = \sigma_{\rm ap}(T^*)$.

It is easily seen from definition of localized SVEP that

$$\lambda \notin \operatorname{acc} \sigma_{\rm ap}(T) \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.4}$$

where $\operatorname{acc} K$ means the set of all accumulation points of $K \subseteq \mathbb{C}$, and if T^* denotes the dual of T then

$$\lambda \notin \operatorname{acc} \sigma_{\rm s}(T) \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{1.5}$$

Remark 1.4. The implications (1.1), (1.2), (1.3), (1.4) and (1.5) are actually equivalences whenever $T \in L(X)$ is semi-Fredholm, or more in general quasi semi-Fredholm (see [1, Chapter 3] and [3]).

Denote by $\operatorname{iso} K$ the set of all isolated points of $K \subseteq \mathbb{C}$. According Berkani and Koliha [15], a bounded operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem* if $\sigma(T) \setminus \sigma_{\rm bw}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\}$. Similarly, a bounded operator $T \in L(X)$ is said to satisfied *generalized a-Weyl's* *theorem* if $\sigma_{\rm ap}(T) \setminus \sigma_{\rm ubw}(T) = E_a(T)$, where $E_a(T) := \{\lambda \in \operatorname{iso}\sigma_{\rm ap}(T) : 0 < \alpha(\lambda I - T)\}$.

Generalized Weyl's theorems have been studied by several authors ([10], [11], [17] and [20]). In this paper we obtain necessary and sufficient conditions for which generalized Weyl's theorems, or generalized a-Weyl's theorem, holds for T. We also consider the case when generalized Weyl's theorems, or generalized a-Weyl's theorem, is transmitted from T to its dual T^*, or to the Hilbert adjoint T' in the case of a Hilbert space operator. Furthermore, we study both generalized Weyl's theorems in the framework of polaroid operators, improving results of [21] concerning Weyl's theorem and a-Weyl's theorem for polaroid operators see ([16].[22]). Our results are applied then to some special classes of operators and, as a consequence, we extend the results of recent papers [20], [29] and [18].

2. Generalized Weyl's theorem

For a bounded operator T, let $\Pi_{00}(T) := \{\lambda \in \sigma(T) : \lambda I - T \text{ is B-Browder}\}$. Observe that in general, $\Pi_{00}(T) \subseteq E(T)$.

A bounded operator $T \in L(X)$ is said to satisfy Browder's theorem if $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$, while T is said to satisfy *generalized Browder's theorem* if $\sigma_{\rm bw}(T) = \sigma_{\rm bb}(T)$.

Theorem 2.1. *If $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies Browder's theorem;*

(ii) *T satisfies generalized Browder's theorem;*

(iii) *T has $SVEP$ at all $\lambda \notin \sigma_{\rm bw}(T)$;*

(iv) *T^* has $SVEP$ at all $\lambda \notin \sigma_{\rm bw}(T)$;*

(v) *T^* satisfies generalized Browder's theorem.*

Proof. A proof the equivalence (i) $\Leftrightarrow$ (ii) may be found in [4]. For the equivalence (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v), see [6]. ∎

Clearly from Theorem 2.1 we have: T or T^* has SVEP implies Browder's theorem holds for T and T^*.

If $T \in L(X)$ let define $E^\sharp(T) := \{\lambda \in \sigma(T) : p(\lambda I - T) = q(\lambda I - T) < \infty\}$.

$E^\sharp(T)$ is exactly the set of poles of the resolvent of T ([25, Proposition 50.2]). Clearly, every pole of the resolvent is an isolated point of the spectrum and it is also an eigenvalue, so $E^\sharp(T) \subseteq E(T)$ for every $T \in L(X)$.

Note that for $T \in L(X)$ satisfies the generalized Weyl's theorem, then T satisfies the generalized Browder's theorem and in general the converse does not hold. Observe that $E^\sharp(T) = \sigma(T) \setminus \sigma_b(T) = \sigma(T) \setminus \sigma_{bb}(T) = \sigma(T) \setminus \sigma_{wb}(T)$, whenever T has the SVEP on $\sigma(T) \setminus \sigma_{wb}(T)$. In consequence, we have the following theorem.

Theorem 2.2. *Let $T \in L(X)$. Then T satisfies the generalized Weyl's theorem if and only if, T satisfies one of the equivalent conditions (i)-(v) of Theorem 2.1 and $E^\sharp(T) = E(T)$.*

Example 2.3. Let $X = \ell^p(\mathbb{N})$, $1 < p < \infty$, and let the unilateral right weighted shift R be defined by

$$R(x_1, x_2, x_3, ...) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, ...) \text{ for all } x = (x_n) \in \ell^p(\mathbb{N}).$$

It is easily seen that R is quasi-nilpotent, $p(R) = \infty$ and hence $E^\sharp(R) = \emptyset$. On the other hand, $\alpha(R) = 1$, so $E(R) = \{0\}$.

Definition 2.4. [24] *Let $T \in L(X)$ and let $d \in \mathbb{N}$. Then T has uniform descent for $n \geq d$, if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geq d$. If in addition $R(T) + N(T^d)$ is closed then T is said to have a topological uniform descent for $n \geq d$.*

Note that every quasi-Fredholm operator has topological uniform descent [16].

Theorem 2.5. *For an operator $T \in L(X)$, the following statements are equivalent:*

(i) $E(T) = E^\sharp(T)$;

(ii) $\sigma_{\mathrm{bb}}(T) \cap E(T) = \emptyset$;

(iii) $\sigma_{\mathrm{bw}}(T) \cap E(T) = \emptyset$;

(iv) $\sigma_{qf}(T) \cap E(T) = \emptyset$;

(v) *If $\lambda \in E(T)$ then $q := q(\lambda I - T) < \infty$ and $(\lambda I - T)^n(X)$ is closed for all $n \geq q$;*

(vi) *If $\lambda \in E(T)$ then $p := p(\lambda I - T) < \infty$ and $(\lambda I - T)^n(X)$ is closed for all $n \geq p$;*

(vii) *If $\lambda \in E(T)$ then there exists $n = n(\lambda) \in \mathbb{N}$ such that $H_0(\lambda I - T) = N(\lambda I - T)^n$.*

(viii) *$\lambda I - T$ has a topological uniform descent for all $\lambda \in E(T)$*

Proof.

(i) $\Rightarrow$ (ii) Clearly, by Corollary 3.4 of [19] we have $\sigma_{\mathrm{bb}}(T) \cap E(T) = \sigma_{\mathrm{d}}(T) \cap E^\sharp(T) = \emptyset$.

(ii) $\Rightarrow$ (iii) Obvious, since $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{bb}}(T)$.

(iii) $\Rightarrow$ (iv) Every semi B-Fredholm operator is quasi-Fredholm, hence $\sigma_{qf}(T) \subseteq \sigma_{\mathrm{bw}}(T)$.

(iv) $\Rightarrow$ (v) By assumption, $\sigma_{qf}(T) \cap E(T) = \emptyset$, hence if $\lambda \in E(T)$ then $\lambda I - T$ is quasi-Fredholm. Now, if $\lambda \in E(T)$ then λ is an isolated point of $\sigma(T) = \sigma(T^*)$, thus both T and T^* have SVEP at λ. By Theorem 3.3 of [19] then $\lambda I - T$ is right Drazin invertible, so $q := q(\lambda I - T) < \infty$ and $(\lambda I - T)^q(X) = (\lambda I - T)^n(X)$ is closed for all $n \geq q$.

(v) $\Rightarrow$ (vi) If $\lambda \in E(T)$ then λ is an isolated point of $\sigma(T)$, hence T has SVEP at λ. By assumption $q = q(\lambda I - T) < \infty$ and $(\lambda I - T)^q(X)$ is closed, hence $\lambda I - T$ is right Drazin invertible, or equivalently, by [19, Theorem 3.3], lower semi B-Browder. Therefore $\lambda I - T$ is quasi-Fredholm and the SVEP of T at λ implies that $\lambda I - T$ is left Drazin invertible, see Theorem 3.2 of [19]. Therefore, $p := p(\lambda I - T) < \infty$. By Theorem 3.3 of [1] then $p(\lambda I - T) = q(\lambda I - T)$ and $(\lambda I - T)^p(X) = (\lambda I - T)^q(X) = (\lambda I - T)^n(X)$ is closed for all $n \geq p$.

(vi) $\Rightarrow$ (vii) Suppose that if $\lambda \in E(T)$ then $p := p(\lambda I - T) < \infty$ and $(\lambda I - T)^n(X)$ is closed for all $n \geq p$. Obviously, $\lambda I - T$ is left Drazin invertible, and since λ is an isolated point of $\sigma(T)$ the SVEP at λ for T entails that $H_0(\lambda I - T) = N(\lambda I - T)^n)$ for some $n \in \mathbb{N}$, see [3, Theorem 2.7].

(vii) $\Rightarrow$ (i) We have only to show that $E(T) \subseteq E^\sharp(T)$. If $\lambda \in E(T)$ then there exists $\nu = \nu(\lambda) \in \mathbb{N}$ such that $H_0(\lambda I - T) = N[(\lambda I - T)^\nu]$. Since λ is an isolated point of $\sigma(T)$ then, by [1, Theorem 3.74], $X = H_0(\lambda I - T) \oplus K(\lambda I - T) = N[(\lambda I -$

$T)^\nu] \oplus K(\lambda I - T)$, hence $(\lambda I - T)^\nu(X) = (\lambda I - T)^\nu(K(\lambda I - T)) = K(\lambda I - T)$. Consequently, $X = N[(\lambda I - T)^\nu] \oplus R[(\lambda I - T)^\nu]$ and this implies that $p(\lambda I - T) = q(\lambda I - T) \leq \nu$, see [1, Theorem 3.6]. Therefore $\lambda \in E^\sharp(T)$, thus the equality $E(T) = E^\sharp(T)$ is proved.

(viii) $\Leftrightarrow$ (i) This equivalence has been proved by Cao in [17]. ∎

Corollary 2.6. *Suppose that $T \in L(X)$ satisfies one of the equivalent conditions (i)-(v) of Theorem 2.1. Then generalized Weyl's theorem holds for T if and only if one of the equivalent conditions (i)-(viii) of Theorem 2.5 holds. In particular, if T or T^* has SVEP then generalized Weyl's theorem holds for T if and only if one of the equivalent conditions (i)-(viii) of Theorem 2.5 holds.*

In the next result, we consider a generalized Weyl' s theorem for T^*. Note that generalized Weyl's theorem is not generally transferred by duality. For instance, if R is the right shift defined in Example 2.3 then its dual $L := R^*$ satisfies a generalized Weyl's theorem, while its dual $L^* = R$ does not satisfy generalized Weyl's theorem.

Theorem 2.7. *(*[1]*). Suppose that T satisfies generalized Weyl's theorem. Then the following conditions are equivalent:*

(i) T^* *satisfies generalized Weyl's theorem;*

(ii) $E(T^*) = E(T)$;

(iii) $E(T^*) \subseteq E(T)$.

A bounded operator $T \in L(X)$ is said to be *polaroid* if $\operatorname{iso}\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent, i.e. $\operatorname{iso}\sigma(T) = E^\sharp(T)$. Every polaroid operator is *isoloid*, i.e. every isolated point of $\sigma(T)$ is an eigenvalue of T. In the proof of Theorem 2.7 we have seen that if λ is a pole of the resolvent of T then λ is a pole of the resolvent of T^*. Since $\operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T^*)$ then follows if T is polaroid then T^* is polaroid.

Theorem 2.8. *Suppose that $T \in L(X)$ is polaroid. If T satisfies Browder's theorem then both T and T^* satisfy generalized Weyl's theorem.*

Proof. By Theorem 2.2 and Theorem 2.1, it suffices to prove $E(T) = E^\sharp(T)$. We need only to prove the following inclusions $E(T) \subseteq E^\sharp(T)$, $E(T^*) \subseteq E^\sharp(T^*)$ and these are clear since T is polaroid if and only if T^* is polaroid. ∎

Let $\mathcal{H}(\sigma(T))$ denote the set of all analytic functions defined on an open neighborhood of $\sigma(T)$ and define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.

Theorem 2.9. *Suppose that $T \in L(X)$ is isoloid and T or T^* has SVEP. If generalized Weyl's theorem holds for T then generalized Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. If T or T^* has SVEP then the spectral mapping theorem holds for $\sigma_{\mathrm{bw}}(T)$, see Theorem 3.4 of [6]. By Theorem 2.1 of [18] then then generalized Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$ (note that in [18] this result is stated in the case of Hilbert space operators, but the proof works also for Banach space operators). ∎

Theorem 2.10. *If $T \in L(X)$ is polaroid and either T or T^* has SVEP then generalized Weyl's theorem holds for $f(T)$ and $f(T^*) = f(T)^*$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. We have seen that Browder's theorem holds for T and T^* whenever either T or T^* have SVEP. Hence, by Theorem 2.8, both T and T^* satisfy generalized Weyl's theorem. Now, T and T^* are isoloid, and hence by Theorem 2.9 $f(T)$, as well as $f(T^*)$, satisfies generalized Weyl's theorem for all $f \in \mathcal{H}(\sigma(T))$. ∎

In the case of operators defined on Hilbert spaces instead of the dual T^* it is more appropriate to consider the Hilbert adjoint T' of $T \in L(H)$. From classical Fredholm theory we have $\sigma_{\rm w}(T') = \overline{\sigma_{\rm w}(T^*)} = \overline{\sigma_{\rm w}(T)}$ and $\sigma_{\rm b}(T') = \overline{\sigma_{\rm b}(T^*)} = \overline{\sigma_{\rm b}(T)}$.

Note that (see [2]) T^* has SVEP if and only if T' has SVEP, so if T' has SVEP then Browder's theorem holds for T and T'.

Theorem 2.11. *Suppose that $T \in L(H)$, H a Hilbert space, is polaroid. If T satisfies Browder's theorem then both T and T' satisfy generalized Weyl's theorem.*

Theorem 2.12. *Suppose that $T \in L(H)$, H a Hilbert space, is polaroid. If either T or T' satisfies SVEP then generalized Weyl's theorem holds for $f(T)$ and $f(T') = f(T)'$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. The SVEP for T or T' entails Browder's theorem for T and T'. Generalized Weyl's theorem for $f(T)$ is clear by Theorem 2.10. To show that generalized Weyl's theorem hold for $f(T')$ observe first that generalized Weyl's theorem holds for T' by Theorem 2.11. The argument of the proof of Theorem 2.11 shows that T' is polaroid, hence isoloid. By [29, Theorem 2.2] then generalized Weyl's theorem holds for $f(T)$ and $f(T') = f(T)'$ for every $f \in \mathcal{H}(\sigma(T))$. ∎

The class of polaroid operators is rather large. In [28] the class of $H(p)$-operators was introduced and defined as the class of all $T \in L(X)$ such that for all $\lambda \in \mathbb{C}$ there exists an integer $p := p(\lambda)$ such that $H_0(T - \lambda I) = N(T - \lambda I)^p$. Property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal, M-hyponormal operators on Hilbert spaces. Furthermore, every multiplier of a commutative semi-simple Banach algebra is $H(1)$, see [1, Theorem 4.33]. A remarkable result of Oudghiri ([28, Theorem 3.4]) shows that, T is $H(p)$ if and only if there exists a function $f \in \mathcal{H}(\sigma(T)$ not identically constant in any component of its domain such that $f(T)$ is $H(p)$, or equivalently that $f(T)$ is $H(p)$ for all $f \in \mathcal{H}(\sigma(T))$. Every $H(p)$-operator T is polaroid [2] and obviously, by (1.3), has SVEP. Therefore, Theorem 2.10 applies to T and, consequently, T satisfies generalized Weyl's theorem. This result may considerably be extended as follows:

Corollary 2.13. *Suppose that $T \in L(X)$ is $H(p)$ on a Banach space X and $f \in \mathcal{H}(\sigma(T)$ is an analytic function not identically constant in any component of its domain. Then $f(T)$ and $f(T)^*$ satisfy generalized Weyl's theorem. If $T \in L(H)$ is a $H(p)$ operator on a Hilbert space H then $f(T') = f(T)'$ satisfies generalized Weyl's theorem.*

Proof. $f(T)$ is a $H(p)$-operator and hence is polaroid and has SVEP. By Theorem 2.10 then $f(T)$ satisfies generalized Weyl's theorem and hence, by Theorem 2.8

(respectively, by Theorem 2.11) also $f(T^*) = f(T)^*$ (respectively, $f(T') = f(T)'$ satisfies generalized Weyl's theorem. ■

We have already observed that every M-hyponormal operator T is a $H(p)$-operator. A bounded operator is said to be *analytically M-hyponormal* (respectively, *algebraically M-hyponormal*) if there exists an analytic function $h \in \mathcal{H}(\sigma(T)$ not identically constant in any component of its domain(respectively, a non trivial polynomial h) such that $h(T)$ is M-hyponormal. Clearly, by Oudhiri's result every algebraically M-hyponormal operator, and more in general every analytically M-hyponormal operator, is $H(p)$, so that Corollary 2.13 extends and subsumes Theorem 4.7 of [20].

A bounded operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ holds for all $x \in X$. Every paranormal operator on a Hilbert space has SVEP, see [8]. An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal*. Every algebraic paranormal operator defined on a Hilbert space is polaroid see [7] and satisfies generalized Weyl's theorem.

Corollary 2.14. *Suppose that $T \in L(H)$, H a Hilbert space, is algebraically paranormal. Then both $f(T)$ and $f(T)'$ satisfy generalized Weyl's theorem for all $f \in \mathcal{H}(\sigma(T)$.*

Proof. Suppose that $h(T)$ is paranormal for some polynomial h. Then $h(T)$ has SVEP and hence, by [1, Theorem 2.40], T has SVEP. Moreover, T is polaroid. By Theorem 2.12 then generalized Weyl's theorem holds for $f(T)$ and $f(T')$. ■

Corollary 2.14 extends Theorem 4.14 of [20] and Theorem 3.1 of [29], while Theorem 2.10 subsumes all these results.

3. Generalized a-Weyl's theorem

In this section by using similar methods to those employed in the previous section, we characterize the bounded linear operators which satisfy generalized a-Weyl's theorem. For a bounded operator $T \in L(X)$, we let

$$\Pi_{00}^a(T) := \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ubb}}(T) = \{\lambda \in \sigma_{ap}(T) : \lambda I - T \text{ is upper semi B-Browder}\}.$$

We have that $\Pi_{00}^a(T) \subseteq E_a(T)$ for any operator $T \in L(X)$.
A bounded operator $T \in L(X)$ is said to satisfy a-Browder's theorem if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, while $T$is said to satisfy *generalized a-Browder's theorem* if $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ubb}}(T)$.

Theorem 3.1. *If $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies a-Browder's theorem;*

(ii) *T satisfies generalized Browder's theorem;*

(iii) *T has $SVEP$ at all $\lambda \notin \sigma_{\mathrm{uw}}(T)$;*

(iv) *T has $SVEP$ at all $\lambda \notin \sigma_{\mathrm{ubw}}(T)$.*

Proof. A proof the equivalence (i) ⇔ (ii) may be found in [4]. For the equivalences (i) ⇔ (iii) and (ii) ⇔ (iii) see [5] and [9]. ■

From Theorem 3.1 follows that if T has SVEP then a-Browder's theorem, or equivalently generalized a-Browder's theorem holds for T. Note that also the SVEP for T^* entails both Browder's theorems, see [9, Corollary 2.10].

Definition 3.2. *Let $T \in L(X)$. $\lambda \in \mathbb{C}$ is said to be a* left pole *of the resolvent of T, if $\lambda \in \sigma_{ap}(T)$ and $\lambda I - T$ is left Drazin invertible.*

In the sequel we set $E_a{}^{\sharp}(T) := \{\lambda \in \sigma_{ap}(T) : \lambda \text{ is a left pole of the resolvent o } T\}$.

Theorem 3.3. *Let $T \in L(X)$. Then T satisfies the generalized a-Weyl's theorem if and only if T satisfies one of the equivalent condition (i)-(iv) of Theorem 3.1 and $E_a(T) = E_a^{\sharp}(T)$.*

Proof. It suffices to prove the equality $\Pi_{00}^a(T) = E_a^{\sharp}(T)$. ∎

Theorem 3.4. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) $E_a(T) = E_a^{\sharp}(T)$;

(ii) $\sigma_{\mathrm{ubb}}(T) \cap E_a(T) = \emptyset$;

(iii) $\sigma_{\mathrm{ubw}}(T) \cap E_a(T) = \emptyset$;

(iv) $\sigma_{qf}(T) \cap E_a(T) = \emptyset$;

(v) for every $\lambda \in E_a(T)$ there exists $d = d(\lambda) \in \mathbb{N}$, such that $H_0(\lambda I - T) = N(\lambda I - T)^d$ and $(\lambda I - T)^n(X)$ is closed for all $n \geq d$;

(vi) for every $\lambda \in E_a(T)$ then $p := p(\lambda I - T) < \infty$ and $(\lambda I - T)^n(X)$ is closed for all $n \geq p$.

Theorem 3.5. *If $T \in L(X)$ then the following statements holds:*

(i) *If T^* has SVEP then $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T)$.*

(ii) *If T has SVEP then $\sigma_{\mathrm{ubw}}(T^*) = \sigma_{\mathrm{bw}}(T^*)$.*

Proof. See [1] ∎

A bounded operator $T \in L(X)$ is said to be *a-polaroid* if iso $\sigma_{\mathrm{ap}}(T) = \emptyset$ or every isolated point of $\sigma_{\mathrm{ap}}(T)$ is a pole of the resolvent, i.e. iso $\sigma_{\mathrm{ap}}(T) = E_a^{\sharp}(T)$. Every a-polaroid operator is *a- isoloid*.

Theorem 3.6. *Suppose that $T \in L(X)$ is a-isoloid. If T or T^* has SVEP and generalized a-Weyl's theorem holds for T then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. If T or T^* has SVEP then the spectral mapping theorem holds for $\sigma_{\mathrm{uw}}(T)$, see Corollary 3.72 of [1]. By Theorem 2.2 of [18] it then follows that a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$. ∎

Theorem 3.7. *Suppose that $T \in L(X)$ is polaroid. Then we have:*

(i) *If T^* has SVEP then generalized a-Weyl theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

(ii) *If T has SVEP then generalized a-Weyl theorem holds for $f(T^*)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Corollary 3.8. *Let T be a Hilbert space operator.*

(i) *If T' is a $H(p)$-operator or is algebraically paranormal then generalized a-Weyl's theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

(ii) *If T is is a $H(p)$-operator or an algebraically paranormal operator then generalized a-Weyl's theorem holds for $f(T')$ for all $f \in \mathcal{H}(\sigma(T))$.*

The result of Corollary 3.8, part (i), in the case of being T' algebraically paranormal, has been proved in [29, Theorem 3.2] by using different methods. Corollary 3.8, part (i), also subsumes Theorem 3.3 of [18], where was considered the case where T' is p-hyponormal or M-hyponormal.

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PARTE II

OPERADORES ASOCIADOS, m-ESTRUCTURAS Y PRODUCTOS GENERALIZADOS

Esta segunda parte del trabajo, de manera similar a la parte anterior, comienza con una visión global del contenido, orientación y propósito de los artículos tratados, y describiendo los aspectos que motivaron el desarrollo de los mismos y los resultados obtenidos.

En topología general, básicamente se estudian la estructura de espacio topológico y las funciones continuas entre estos espacios. En la formulación y estudio de las propiedades asociadas a un espacio topológico abstracto, así como también de las funciones continuas entre éstos, juegan un papel preponderante los conjuntos abiertos. También entran en juego los conjuntos cerrados, pero éstos pueden verse como los duales de los conjuntos abiertos con respecto a las operaciones conjuntistas de complementación, unión e intersección. En el año 1963, N. Levine ([16]), introduce los conjuntos semiabiertos que son una clase más amplia que la de los conjuntos abiertos de un espacio. Después de los trabajos de Levine, el interés de muchos matemáticos dedicados a la topología se centró en la generalización de conceptos topológicos y diversas formas de continuidad, utilizando conjuntos semiabiertos en lugar de conjuntos abiertos. S. Kasahara, en el año 1979 ([15]), da el concepto de operador asociado a una topología lo que permite definir nuevas clases aún más abstractas de conjuntos e introducir propiedades generalizadas de separación, continuidad, compacidad y otras nociones derivadas de las nociones topológicas clásicas, pero presentadas ahora en un nuevo y más amplio contexto. Si bien esto originó que en las últimas décadas se hayan venido realizando intensos trabajos de investigación en lo relativo a generalización de nociones topológicas vía operadores; en particular, en la literatura habían aparecido muy pocos resultados relacionados con operadores que actuaban sobre el espacio producto. Salvo el trabajo de T. Fukutake ([11]), que trata el caso de dos factores, esta situación prácticamente no había sido estudiada. En este sentido, el primer artículo que contiene esta parte del trabajo titulado "A Tychonoff theorem for α-compactness and some applications", de C. Carpintero, E. Rosas y J. Sanabria, publicado en la revista Indian Journal of Mathematics, en el año 2007, aborda esta situación y en el se logra demostrar una versión del Teorema de Tchynoff para la α-compacidad, cualquiera sea el operador α asociado a la topología producto compatible con los operadores presentes en los factores.

Centrando su atención en algunas propiedades de diversas clases de conjuntos derivados de la noción de operador asociado de Kasahara; H. Maki en 1996 ([17]), da un enfoque más axiomático a la generalización de nociones topológicas a través de operadores, introduciendo el concepto de m-estructura o estructura minimal. En este contexto se introduce el segundo artículo de esta parte, titulado "Minimal structures and separations properties", de C. Carpintero, E. Rosas y M. Salas publicado en la revista International Journal of Pure and Applied Mathematics el año 2007, en el cual se logra proporcionar una imagen global del comportamiento de las propiedades de separación, mediante la noción de m-estructura. Lo que permite reducir a un sólo marco teórico muchos de los trabajos publicados aisladamente en este campo. En esta misma dirección, también se introduce el artículo "Conjuntos m_X-cerrados generalizados" de M. Salas, C. Carpintero y E. Rosas, publicado en la revista Divulgaciones Matemáticas el año 2007, en el que se generalizan resultados relativos a conjuntos g-cerrados y axiomas bajos de separación.

De manera similar a Maki; Á. Császár en el año 2005 ([7]), introduce la noción de topología generalizada (abreviada GT). Cabe señalar que la sutil diferencia entre las nociones de m-estructura y topología generalizada, esta determinada por una propiedad conocida como la propiedad (B) de Maki ([17],[18]), también conocida simplemente como la propiedad de Maki ([17],[18]). No obstante, en ambos contextos, en el artículo titulado "Inadmissible families and product of generalized topologies" de C. Carpintero, E. Rosas, O. Özbakir y J. Salazar, publicado en la revista International Mathematical Forum el año 2010, se presenta un nuevo planteamiento que permite analizar el comportamiento del producto arbitrario de topologías generalizadas (respectivemente, m-espacios), y en el cual no sólo se generalizan los resultados obtenidos por Á. Császár en [8], si no que además se logra demostrar que muchas formas generalizadas de compacidad se transmiten de los factores al producto y viceversa.

2.1. UN TEOREMA DE TYCHONOFF PARA LA α-COMPACIDAD

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A TYCHONOFF THEOREM FOR α-COMPACTNESS AND SOME APPLICATIONS

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In this article, we have introduced a new class of associated operators on the product topology on which each factor of the product space has an associated operator to the respective topology and we will prove an analogue to the Tychonoff theorem for α-compactness. Moreover, we investigate the relationship between the new operators on the product topology and the operators associated to each factor, and we study some classes of functions and their connections with the notions mentioned.

1. Introduction

Given a topological space (X, τ), along with the classical concepts of open set and closed sets, there are classes of sets that have been studied: semi open sets ($A \subset Cl(Int(A))$), pre-open sets ($A \subseteq Int(Cl(A))$), pre-semi open sets ($A \subseteq Int(Cl(Int(A)))$). These sets have been studied by many mathematicians, as is the case of: Levine [5], Bhattacharyya and

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Balachandran [1], in order to introduce the generalized properties of separation, continuity and related notions. We can observe that the above sets and its compositions are classical examples of operators. We define $\alpha : P(X) \to P(X)$ as an operator associated with τ on X if $U \subseteq \alpha(U)$ for all $U \in \tau$. Rosas, Carpintero and Vielma in [9], using this notion, introduced the α-semi-open sets (A is an α-semi-open set if $U \subseteq A \subseteq \alpha(U)$, for some $U \in \tau$) and α-semi-closed sets (A is an α-semi-closed if $X \setminus A$ is an α-semi-open set) and have shown more general notions and results by the mentioned authors. In the literature are known a few results about product spaces when in its factors the notions of semi open sets, pre-open sets, pre-semi open sets, etc. Moreover the behaviour of the product space, when each of its factors satisfies some specific conditions with respect to the associated operators has not been studied. In this paper, we study a class of operators on the product space where each factor has the property of α- compactness. Some other results related to this class of operators are also obtained.

2. Preliminary

In this section, we mention the terminology and some basic results that we use throughout this article.

Definition 2.1. Let (X, τ) be a topological space. The operator $\alpha : P(X) \to P(X)$ is said to be an operator associated to τ if $U \subseteq \alpha(U)$, for all $U \in \tau$.

Definition 2.2. Let (X, τ) be a topological space and $\alpha : P(X) \to P(X)$ be an operator associated to τ. If $\alpha(U) \subseteq \alpha(V)$, for all $U \subseteq V$, then α is said to be a monotone operator.

Remark 2.1. Given an operator $\alpha : P(X) \to P(X)$ associated to τ, the image $\alpha(S)$ of a subset $S \subseteq X$ is a subset of X. The image of a collection $\mathcal{F}$ of subsets of X by the operator α, denoted by $\alpha(\mathcal{F})$, is the collection

$$\alpha(\mathcal{F}) = \{\alpha(U) : U \in \mathcal{F}\}.$$

In a natural way, we can define the image of a family $\Im$, where its members are collections of subsets of X, i.e $\Im \subseteq P(P(X))$, by the operator α as follows:

$$\alpha(\Im) = \{\alpha(\mathcal{F}) : \mathcal{F} \in \Im\},$$

where $\alpha(\mathcal{F}) = \{\alpha(U) : U \in \mathcal{F}\}$, for each collection of subsets $\mathcal{F} \in \Im$. Observe that $\alpha(\Im)$ defined in this way is a family whose members are collections of subsets of X.

Abusing the notation and without danger of confusion, according to the context, $\alpha(S)$ (resp. $\alpha(\mathcal{F})$, $\alpha(\Im)$) denote a subset of X (resp. a collection of subsets of X, a family whose members are collections of subsets of X).

The following example shows the form that we are going to use in the above notation.

Example 2.1. Let $\alpha = Cl$ be the closure operator associated with the usual topology in $\mathbf{R}$. Consider: $S = (a, +\infty)$, $\mathcal{F} = \{(a, +\infty) : a \in \mathbf{R}\}$ and $\Im = \{\mathcal{F}_a : a \in \mathbf{R}\}$, with $\mathcal{F}_a = \{(a, b) : b > a\}$, for each $a \in \mathbf{R}$. Then:

(i) $\alpha(S) = [a, +\infty)$,

(ii) $\alpha(\mathcal{F}) = \{[a, \infty) : a \in \mathbf{R}\}$,

(iii) $\alpha(\Im) = \{\{[a, b] : b > a\} : a \in \mathbf{R}\}$.

Definition 2.3. Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces and for each $i \in I$, $\alpha_i : P(X_i) \to P(X_i)$ denote the operator associated to τ_i. We say that the operator $\rho : P(X) \to P(X)$ associated to a product topology, where $X = \prod_{i\in I} X_i$ is compatible with the α_i, $i \in I$, if for each basic open set $< U_{i_1}, U_{i_2}, ..., U_{i_n} >$ in $X = \prod_{i\in I} X_i$ we have

$$\rho(< U_{i_1}, U_{i_2}, ..., U_{i_n} >) =< \alpha_{i_1}(U_{i_1}), \alpha_{i_2}(U_{i_2}), ..., \alpha_{i_n}(U_{i_n}) >,$$

where

$$\begin{aligned} < U_{i_1}, U_{i_2}, ..., U_{i_n} > &= (\prod_{j=1}^{n} X_{i_j}) \times (\prod_{k\neq i_j} X_k) \\ &= \bigcap_{j=1}^{n} p_{i_j}^{-1}(U_{i_j}) \end{aligned}$$

and

$$\begin{aligned} < \alpha_{i_1}(U_{i_1}), \alpha_{i_2}(U_{i_2}), ..., \alpha_{i_n}(U_{i_n}) > &= (\prod_{j=1}^{n} \alpha_{i_j}(U_{i_j}) \times (\prod_{k \neq i_j} X_k) \\ &= \bigcap_{j=1}^{n} p_{i_j}^{-1}(\alpha_{i_j}(U_{i_j})), \end{aligned}$$

and $p_i : X \to X_i$, $i \in I$, is the i^{th} projection.

Remark 2.2. In general $\rho = \prod_{i \in I} \alpha_i$, given by $\rho(\prod_{i \in I} A_i) = \Pi \alpha_i(A_i)$ is not an operator associated to a product topology, because the empty set has many representations. Thus ρ does not represent a function.

Example 2.2. If in Definition 2.3, for each $i \in I$, we define

$$\alpha_i = Cl \text{ (resp. } Cl(Int), Int(Cl), Int(Cl(Int))).$$

Then $\rho = Cl$ (resp. $Cl(Int)$,$Int(Cl)$,$Int(Cl(Int))$) is in each case an operator on X compatible with the α_i.

The following example shows that there exists many operators on the product space which do not equal to the operator defined in the Example 2.2. But they are compatible with the given operators.

Example 2.3. Let $f_i : X_i \to X_i$, $i \in I$, be functions. Define $f : X \to X$, where $X = \prod_{i \in I} X_i$, as follows:

$$f(x) = (f_i(x_i))_{i \in I} \text{ for each } x = (x_i)_{i \in I}.$$

Observe that

$$f(\prod_{i \in I} A_i) = \prod_{i \in I} f_i(A_i),$$

for any $A_i \subseteq X_i$, and

$$f^{-1} f(\prod_{i \in I} A_i) = \prod_{i \in I} f_i^{-1}(f_i(A_i)).$$

If we consider, for each $i \in I$, τ_i a topology on X_i, then $\alpha_i(A_i) = f_i^{-1}(f_i(A_i))$ and $\rho(A) = f^{-1}(f(A))$, are operators associated with the topology τ_i and the product topology, respectively. If each $f_i : X_i \to X_i$ is surjective, then ρ is compatible with the operators α_i. Observe that α_i and ρ are monotone operators.

Now we study the structure of the ρ-semi-open sets in the product space $X = \prod_{i\in I} X_i$, in the case when ρ is compatible with the operators associated with the topology on each space X_i.

LEMMA 2.1. *Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces with associated operators α_i to τ_i, for each $i \in I$. Suppose $\rho : P(X) \to P(X)$, where $X = \prod_{i\in I} X_i$, is a monotone and compatible operator with the α_i such that $\rho(\emptyset) = \emptyset$. If $\emptyset \neq \prod_{i\in I} A_i$, $A_i \subseteq X_i$, is a ρ-semi-open set in X, then A_i is α_i-semi-open set in X_i for each $i \in I$.*

PROOF. Suppose that $\emptyset \neq \prod_{i\in I} A_i$ is a ρ-semi-open set in X. Then there exists an open set $U \subseteq X$ such that $U \subseteq \prod_{i\in I} A_i \subseteq \rho(U)$. It is clear that $U \neq \emptyset$ because if $U = \emptyset$. Then $\emptyset \neq \prod_{i\in I} A_i \subseteq \rho(\emptyset) = \emptyset$, this is impossible by the hypothesis. Let $p_j : X \to X_j$ be the j^{th} projection. Then $p_j(U) \subseteq p_j(\prod_{i\in I} A_i) = A_j$. Hence $p_j(U) \subseteq A_j$, for each $j \in I$. On the other hand, for all $j \in I$ we have

$$U \subseteq \prod_{i\in I} p_j(U) \subseteq \quad < p_j(U) > .$$

By hypothesis ρ is monotone and compatible with each α_i. Since $p_j : X \to X_j$ is an open map, we obtain that

$$\prod_{i\in I} A_i \subseteq \rho(U) \subseteq \rho(< p_j(U) >) = \quad < \alpha_j(p_j(U)) > .$$

This implies that $A_j \subseteq \alpha_j(p_j(U))$ for each $j \in I$.

By the above argument we see that there exists an open set $p_j(U) \subseteq X_j$ which satisfies

$$p_j(U) \subseteq A_j \subseteq \alpha_j(p_j(U)),$$

from which we conclude that each A_j is an α_j-semi-open set in X_j.

COROLLARY 2.2 *Under the hypothesis of Lemma 2.1, if the product $\prod_{i\in I} A_i$, is a nonempty proper subset and ρ-semi-open set of X, then there exists a finite subset $\{i_1\ i_2, ..., i_n\} \subseteq I$ such that the α_i-semi-open sets A_i are distinct from X_i, for each $i \in \{i_1, i_2, ..., i_n\}$.*

Proof. By hypothesis there exists an open set $\emptyset \neq U \subseteq X$ such that

$$U \subseteq \prod_{i \in I} A_i \subseteq \rho(U).$$

Consequently there exists a point $x \in U$ and a basic open set in X

$$< U_{i_1}, U_{i_2}, ..., U_{i_n} >,$$

such that

$$x \in \quad < U_{i_1}, U_{i_2}, ..., U_{i_n} > \quad \subseteq U \subseteq \prod_{i \in I} A_i \subseteq \rho(U).$$

It follows that $x \in < U_{i_1}, U_{i_2}, ..., U_{i_n} > \subseteq \prod_{i \in I} A_i$. But this implies that $X_j = A_j$, for each $j \notin \{i_1, i_2, ..., i_n\}$.

The following theorem generalizes the results obtained by Bhattacharyya and Lahiri [1] in the context of associated operators.

Theorem 2.3. *Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces with operators $\alpha_i : P(X_i) \to P(X_i)$ associated with τ_i, for each $i \in I$, and $\rho : P(X) \to P(X)$, where $X = \prod_{i \in I} X_i$, be a monotone and compatible operator with the α_i such that $\rho(\emptyset) = \emptyset$. Then $< A_{i_1}, A_{i_2}, ..., A_{i_n} >$ is ρ-semi-open $\Leftrightarrow$ A_{i_j} is α_{i_j}-semi-open $j = 1, ..., n$.*

Proof. (*Sufficiency*). It follows from Lemma 2.1.

(*Necessity*). Let A_{i_j}, $A_{i_j} \neq X_{i_j}$, be a α_{i_j}-semi-open set in X_{i_j} for each $j \in \{1, 2, ..., n\}$. By hypothesis, there exist open sets $U_{i_j} \subseteq X_{i_j}$ such that $U_{i_j} \subseteq A_{i_j} \subseteq \alpha_{i_j}(U_{i_j})$, for $j = 1, 2, ..., n$. Note that from $U_{i_j} \subseteq A_{i_j} \neq X_{i_j}$, we obtain that $U_{i_j} \neq X_{i_j}$ for each $j \in \{1, 2, ..., n\}$. Therefore

$$\begin{aligned} < U_{i_1}, U_{i_2}, ..., U_{i_n} > &\subseteq < A_{i_1}, A_{i_2}, ..., A_{i_n} > \\ &\subseteq < \alpha_{i_1}(U_{i_1}), \alpha_{i_2}(U_{i_2}), ..., \alpha_{i_n}(U_{i_n}) >, \end{aligned}$$

from which we obtain that

$$< U_{i_1}, U_{i_2}, ..., U_{i_n} > \subseteq < A_{i_1}, A_{i_2}, ..., A_{i_n} > \subseteq \rho(< U_{i_1}, U_{i_2}, ..., U_{i_n} >).$$

Thus $< A_{i_1}, A_{i_2}, ..., A_{i_n} >$ is ρ-semi-open.

Corollary 2.4. *Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces with operators $\alpha_i : P(X_i) \to P(X_i)$ associated with τ_i, for each*

$i \in I$, and let $\rho : P(X) \to P(X)$, where $X = \prod_{i\in I} X_i$, be a monotone and compatible operator with the α_i such that $\rho(\emptyset) = \emptyset$. Then:

$$\rho - sCl(\prod_{i\in I} A_i) \subseteq \prod_{i\in I} \alpha_i - sCl(A_i),$$

where $\rho - sCl$ and $\alpha_i - sCl$ are defined in [3].

Now, we introduce the (α, β)-irresolute functions that will be useful in the sequel.

DEFINITION 2.4. Let (X, τ) and (Y, σ) be two topological spaces with operators α, β associated with the topology τ and σ, respectively. Then $f : X \to Y$ is said to be an (α, β)-irresolute if $f^{-1}(V)$ is α-semi-open set in X for each β-semi-open set V in Y.

REMARK 2.3. Observe that:

(i) If $f : (X, \tau, \alpha) \to (Y, \sigma, \beta)$ is a (α, β)-irresolute function and $g : (Y, \sigma, \beta) \to (Z, \theta, \gamma)$ is a (β, γ)-irresolute function, then the composition $gf : (X, \tau, \alpha) \to (Z, \theta, \gamma)$ is a (α, γ)-irresolute function.

(ii) f is a (Cl_X, id)-irresolute function $\Leftrightarrow$ f is a semi-continuous map (in the sense of Levine [5]).

(iii) f is a $(id_{P(X)}, id_{P(Y)})$-irresolute function $\Leftrightarrow$ f is a continuous map.

(iv) f is a (Int_X, Int_Y)-irresolute function $\Leftrightarrow$ f is an open map.

THEOREM 2.5. *Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces with operators, α_i associated with each topology τ_i* and *$\rho : X \to X$, where $X = \prod_{i\in I} X_i$, be a compatible operator with the α_i. Then the i^{th} projection $p_i : X \to X_i$ is a (ρ, α_i)-irresolute function for each $i \in I$.*

PROOF. Let $A_i \neq X_i$ be a α_i-semi-open set in X_i. Then there exists an open set $U_i \in \tau_i$ such that $U_i \subseteq A_i \subseteq \alpha_i(U_i)$. Consequently

$$< U_i >= p_i^{-1}(U_i) \subseteq p_i^{-1}(A_i) \subseteq p_i^{-1}(\alpha_i(U_i)) =< \alpha_i(U_i) >= \rho(< U_i >).$$

Therefore $p_i^{-1}(A_i)$ is a ρ-semi-open set.

In the case where $A_i = X_i$, $p_i^{-1}(A_i) = \Pi_{i\in I} X_i$ is a ρ semi-open set.

COROLLARY 2.6. *Let $C = \{(X_i, \tau_i) : i \in I\} \cup \{(Y, \sigma)\}$ be a collection of topological spaces with operators α_i associated with τ_i, and for each $i \in I$,*

$f_i : Y \to X_i$ be functions. If $\rho : P(X) \to P(X)$, $X = \prod_{i \in I} X_i$, is a monotone and compatible operator with the α_i such that $\rho(\emptyset) = \emptyset$, and $f : Y \to X$ is defined by $f(y) = (f_i(y))_{i \in I}$, if f is a $(id_{P(Y)}, \rho)$-irresolute function, then each f_i is a $(id_{P(Y)}, \alpha_i)$-irresolute function.

3. α-Compactness in the Product Space

In this section, we shall study the relationship between the generalized notions of a compatible operator in the product space with the generalized notions relative to each factor.

Definition 3.1. Let (X, τ) be a topological space and $\alpha : P(X) \to P(X)$ be an operator associated with τ. Then X is said to be α-compact space if every open cover $\mathcal{C}$ of X contains a finite subcollection $\{C_1, ..., C_n\} \subseteq \mathcal{C}$ such that $X = \bigcup_{k=1}^{n} \alpha(C_k)$.

Definition 3.2. Let X be a nonempty subset. A collection $\mathcal{F}$ of subsets of X is said to be finitely inadmissible family, (briefly) f.i, if no finite subcollection of $\mathcal{F}$ covers X.

The following lemma is a classical result. We use the axiom of choice to prove this result.

Lemma 3.1. *Let $\mathcal{F}$ be a finitely inadmissible family of subsets of X. The following assertions hold:*

(i) *There is a finitely inadmissible family $\mathcal{F}^*$ of subsets of X such that $\mathcal{F} \subseteq \mathcal{F}^*$. Moreover $\mathcal{F}^*$ is maximal with respect to the partial order*

$$\mathcal{C} \prec \mathcal{C}' \Leftrightarrow \mathcal{C} \subset \mathcal{C}'$$

defined on the set $\Im$ of all finitely inadmissible family of subsets of X containing $\mathcal{F}$.

(ii) *If $S_1 \cap S_2 \cap ... \cap S_n \in \mathcal{F}^*$, then $S_k \in \mathcal{F}^*$ for some $1 \leq k \leq n$.*

(iii) *If $S \notin \mathcal{F}^*$ and $S \subseteq S'$, then $S' \notin \mathcal{F}^*$.*

Proof. (i) $\{\mathcal{F}\} \subseteq \Im$ is a simple order set. Now using the maximal principle, there exists a maximal simply ordered subset $\Im' \subseteq \Im$ such that $\{\mathcal{F}\} \subseteq \Im'$, so $\mathcal{F} \in \Im'$. Now we define $\mathcal{F}^* = \bigcup_{\mathcal{C} \in \Im'} \mathcal{C}$. Clearly $\mathcal{F} \subseteq \mathcal{F}^*$. On the other hand, if there exists a finite collection $\{S_1, S_2, ..., S_n\} \subseteq \mathcal{F}^*$ such that $X = \bigcup_{k=1}^{n} S_k$, then $S_k \in \mathcal{C}_k$ and $\mathcal{C}_k \in \Im'$ for each $k = 1, 2, ..., n$.

But $\Im'$ is simply ordered set, so there exists some $1 \leq j \leq n$, such that $C_k \subseteq C_j$, for all $k = 1, 2, ..., n$. This implies that $S_k \in C_j$, for $k = 1, 2, ..., n$, and $X = \bigcup_{k=1}^{n} S_k$; but it is impossible because C_j is a finitely inadmissible family. Consequently, $\mathcal{F}^*$ is a finitely inadmissible family. Finally, we observe that from the definition of $\mathcal{F}^*$, it follows that $C \subseteq \mathcal{F}^*$ for any $C \in \Im'$. Therefore $\Im' \cup \{\mathcal{F}^*\}$ is simply ordered. From the maximality of $\Im'$, it follows that $\mathcal{F}^* \in \Im'$.

(ii) and (iii), are direct consequences of the definition of $\mathcal{F}^*$.

Remark 3.1. Suppose that (X, τ) is a topological space, α an operator associated with τ, $\mathcal{F}$ is a collection of subsets of X such that $\alpha(\mathcal{F}) = \{\alpha(U) : U \in \mathcal{F}\}$ a finitely inadmissible family, and $\Re$ denotes the family of all collections of subsets of X that are finitely inadmissible and contain $\alpha(\mathcal{F})$. Then using a similar argument as in the proof of Lemma 3.1(i), there exists a maximal simply ordered subset $\Re' \subseteq \Re$ such that $\{\alpha(\mathcal{F})\} \subseteq \Re'$. If ${\mathcal{F}^*}_\alpha$ denotes the finitely inadmissible family of subsets of X which is maximal with respect to the inclusion on $\Re$ and contain $\alpha(\mathcal{F})$, defined in Lemma 3.1(i), then

$$\mathcal{F}^*{}_\alpha = \bigcup_{\mathcal{G} \in \Re'} \mathcal{G}.$$

Using the notation of Lemma 3.1, we obtain the following result.

Theorem 3.2. *If $\mathcal{F}$ and $\alpha(\mathcal{F})$ are finitely inadmissible collections of subsets of X, then $\alpha(\mathcal{F}^*) \subseteq \mathcal{F}^*{}_\alpha$.*

Proof. From Lemma 3.1 and Remark 3.1, we obtain that

$$\mathcal{F}^* = \bigcup_{C \in \Im'} C$$

and

$$\mathcal{F}^*{}_\alpha = \bigcup_{\mathcal{G} \in \Re'} \mathcal{G}.$$

Now it follows that

$$\alpha(\mathcal{F}^*) = \alpha(\bigcup_{C \in \Im'} C) = \bigcup_{C \in \Im'} \alpha(C) = \bigcup_{C' \in \alpha(\Im')} C'.$$

Since $\{\mathcal{F}\} \subseteq \Im'$, then $\{\alpha(\mathcal{F})\} \subseteq \alpha(\Im')$. But $\alpha(\Im')$ is simply ordered, since $\Im'$ is simply ordered. Using the maximality of $\Re'$, we then obtain that $\alpha(\Im') \subseteq \Re'$. Therefore $\alpha(\mathcal{F}^*) \subseteq \mathcal{F}^*{}_{\alpha}$.

The following lemma is a generalized version of the Alexander Lemma ([10]), for α-compact spaces.

Lemma 3.3. *Let (X, τ) be a topological space and $\alpha : P(X) \to P(X)$ an operator associated with τ and $\mathcal{S}$ be a subbase for τ. Then X is an α-compact space if and only if each covering $\mathcal{C}$* of *X by elements of $\mathcal{S}$ contains a finite subcollection $\{C_1, C_2, ..., C_n\}$ such that $X = \bigcup_{k=1}^{n} \alpha(C_k)$.*

Proof. (*Sufficiency*). It is an immediate consequence of Definition 3.1. (*Necessity*). Suppose that X is not α-compact space. Then there exists an open cover $\mathcal{C}$ of X such that the collection of subsets

$$\alpha(\mathcal{C}) = \{\alpha(\mathrm{C}) : \mathrm{C} \in \mathcal{C}\},$$

does not contain any finite subcollection that covers X. Observe first that the covering $\mathcal{C}$ does not contain any finite subcollection $\{C_1, C_2, ..., C_n\}$ that covers X. In fact, if

$$X = \bigcup_{k=1}^{n} C_k \subseteq \bigcup_{k=1}^{n} \alpha(C_k),$$

then $\alpha(\mathcal{C})$ contains a finite subcollection $\{\alpha(C_1), \alpha(C_2), ..., \alpha(C_n)\}$ that covers X, and this is impossible. Using this fact, we have that $\mathcal{C}$ and $\alpha(\mathcal{C})$ are finitely inadmissible family of subsets of X. On the other hand, for each point $x \in X$ there exists an open set $U_x \in \mathcal{C}$ and elements $S_1^{(x)}, S_2^{(x)}, ..., S_n^{(x)} \in \mathcal{S}$, such that $x \in S_1^{(x)} \cap S_2^{(x)} \cap ... \cap S_n^{(x)} \subseteq U_x$. If $\mathcal{C}^*$ is the maximal finitely inadmissible family corresponding to the collection $\mathcal{C}$, given in Lemma 3.1, it then follows that

$$x \in S_1^{(x)} \cap S_2^{(x)} \cap ... \cap S_n^{(x)} \subseteq U_x \in \mathcal{C} \subseteq \mathcal{C}^*.$$

Now, using this fact and Lemma 3.1(iii), we have $S_1^{(x)} \cap S_2^{(x)} \cap ... \cap S_n^{(x)} \in \mathcal{C}^*$. Again by using Lemma 3.1(ii), there exists element $S_k^{(x)} \in \mathcal{C}^*$, $1 \leq k \leq n$, such that $x \in S_k^{(x)}$. Therefore, for each $x \in X$ there exists $S_k^{(x)} \in \mathcal{C}^* \cap \mathcal{S}$ such that $x \in S_k^{(x)}$, and hence $\mathcal{C}^* \cap \mathcal{S}$ is a collection of subbasics elements that covers X. Observe that $\alpha(\mathcal{C}^* \cap \mathcal{S}) \subseteq \alpha(\mathcal{C}^*)$ and by Theorem 3.2 $\alpha(\mathcal{C}^*) \subseteq \mathcal{C}^*{}_{\alpha}$. Therefore, $\alpha(\mathcal{C}^* \cap \mathcal{S}) \subseteq \mathcal{C}^*{}_{\alpha}$, we will have that $\alpha(\mathcal{C}^* \cap \mathcal{S})$ is finitely inadmissible and does not exist a finite subcollection $\{S_1, S_2, ..., S_n\} \subseteq \mathcal{C}^* \cap \mathcal{S}$ such that $\{\alpha(S_1), \alpha(S_2), ..., \alpha(S_n)\}$ covers X.

The following theorem generalizes the classical Tychonoff theorem, which is obtained from the special case $\alpha = id_{P(X)}$, also generalizes a result obtained by Fukutake in [4].

THEOREM 3.4. *Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces with operators $\alpha_i : P(X_i) \to P(X_i)$ associated with τ_i, for each $i \in I$, and $\rho : P(X) \to P(X)$, where $X = \prod_{i \in I} X_i$, be a compatible operator with the α_i. If $X = \prod_{i \in I} X_i \neq \emptyset$, then X is a ρ-compact space $\Leftrightarrow$ each X_i is an α_i-compact space.*

PROOF. (*Sufficiency*). Let $\mathcal{C}^i = \{U^i_\lambda : \lambda \in \Lambda_i\}$ be an open cover of X_i, $i \in I$. Then the collection $\{< U^i_\lambda >: \lambda \in \Lambda_i\}$, is an open cover of X. By hypothesis, X is a ρ-compact space, so there exists a finite subcollection $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset \Lambda_i$ such that $X = \bigcup_{k=1}^n \rho(< U^i_{\lambda_k} >)$. But ρ is compatible with the operators α_j, $j \in I$, so

$$X = \bigcup_{k=1}^{n} < \alpha_i(U^i_{\lambda_k}) > .$$

Taking the i^{th} projection, we obtain that:

$$X_i = p_i^{-1}(X) = p_i^{-1}(\bigcup_{k=1}^{n}(< \alpha_i(U^i_{\lambda_k}) >)$$

$$= \bigcup_{k=1}^{n} p_i^{-1}(< \alpha_i(U^i_{\lambda_k}) >) = \bigcup_{k=1}^{n} \alpha_i(U^i_{\lambda_k}).$$

Therefore

$$X_i = \bigcup_{k=1}^{n} \alpha_i(U^i_{\lambda_k}),$$

and X_i is an α_i-compact space.

(*Necessity*). Suppose that each X_i is an α_i-compact space and X is not a ρ-compact space. Using Lemma 3.3, there exists a covering $\mathcal{C}$ of X, by subbasic elements in the product topology on X, and there does not exist any finite subcollection $\{C_1, C_2, ..., C_n\} \subseteq \mathcal{C}$ such that $\{\rho(C_1), \rho(C_2), ..., \rho(C_n)\}$ covers X. For each $i \in I$, consider

$$\mathcal{C}^i = \{U^i : U^i \in \tau_i \text{ and } p_i^{-1}(U^i) =< U^i >\in \mathcal{C}\}.$$

We claim that, $\mathcal{C}^i$ is not a covering of X_i, for each $i \in I$; In fact, suppose that $\mathcal{C}^i$ is a covering of X_i, then the α_i-compactness of X_i implies the existence of a finite subcollection $\{U_1^i, U_2^i, ..., U_n^i\} \subseteq \mathcal{C}^i$ such that

$$X_i = \bigcup_{k=1}^{n} \alpha_i(U_k^i)$$

Therefore

$$\begin{aligned} X &= p_i^{-1}(X_i) \\ &= p_i^{-1}(\bigcup_{k=1}^{n} \alpha_i(U_k^i)) \\ &= \bigcup_{k=1}^{n} p_i^{-1}(\alpha_i(U_k^i)) \\ &= \bigcup_{k=1}^{n} < \alpha_i(U_k^i) > \\ &= \bigcup_{k=1}^{n} \rho(< U_k^i >). \end{aligned}$$

Hence $X = \bigcup_{k=1}^{n} \rho(< U_k^i >)$, and this is impossible, because $< U_k^i >\in \mathcal{C}$ for each $k = 1, 2, ..., n$. Therefore $X_i \not\subseteq \bigcup\{U_k^i : U_k^i \in \mathcal{C}^i\}$ as claimed. Consequently, there exists a point $z_i \in X_i \setminus \bigcup\{U_k^i : U_k^i \in \mathcal{C}^i\}$ for each $i \in I$. Define $z = (z_i)_{i \in I}$. Since $\mathcal{C}$ covers X there exists a subbasic element $< U^j >\in \mathcal{C}$, such that $z \in < U^j >$. This implies that $p_j(z) = z_j \in U_j$, which is impossible, because $U^j \in \mathcal{C}^j$.

COROLLARY 3.5. *Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces with operators $\alpha_i : P(X_i) \to P(X_i)$ associated with τ_i, for each $i \in I$, and $\rho : P(X) \to P(X)$, where $X = \prod_{i \in I} X_i$, be a compatible operator with the α_i such that $\rho(\emptyset) = \emptyset$. Then,*

$$X_i \textit{ is } \alpha_i\textit{-semi } T_k \Rightarrow \Pi_{i \in I} X_i \textit{ is } \rho\textit{-semi } T_k,$$

for each $k = 0, 1, 2$.

4. Applications in the Finite Case

THEOREM 4.1. *Let* (X, τ), (Y, σ), (Z, θ) *be three topological spaces with operators* α, β, γ *associated with* τ, σ *and* θ, *respectively. Let* $\rho : P(X \times Y) \to P(X \times Y)$ *be a monotone operator which is compatible with* α *and* β. *If* $f : X \to Z$ *and* $g : Y \to Z$ *are* (α, θ) *and* (β, θ)*-irresolute functions, respectively, and* (Z, θ) *is a* θ-semi T_2 [9], *then the following set*

$$\{(x, y) \in X \times Y : f(x) = g(y)\}$$

is ρ*-semi-closed in* $X \times Y$.

PROOF. Let $A = \{(x, y) \in X \times Y : f(x) = g(y)\}$. If $(x, y) \notin A$, then $f(x) \neq g(y)$ in Z. By the hypothesis (Z, ρ) is a θ-semi T_2, so there exist open sets W_x, W_y in Z such that $f(x) \in W_x$, $g(y) \in W_y$ and $\theta(W_x) \cap \theta(W_y) = \emptyset$. But each open set in Z is θ-semi-open and since f is (α, θ)-irresolute function, follows that $f^{-1}(W_x)$ is an α-semi-open and $x \in f^{-1}(W_x)$. In the same way $g^{-1}(W_y)$ is β-semi-open and $y \in g^{-1}(W_y)$. We claim that $(f^{-1}(W_x) \times g^{-1}(W_y)) \cap A = \emptyset$. In fact if $(f^{-1}(W_x) \times g^{-1}(W_y)) \cap A \neq \emptyset$ then there exist $(u, v) \in (f^{-1}(W_x) \times g^{-1}(W_y)) \cap A$ and hence $f(u) \in W_x$, $g(v) \in W_y$, and $f(u) = g(v)$. From this it follows that $W_x \cap W_y \neq \emptyset$, which is impossible since $W_x \cap W_y \subseteq \theta(W_x) \cap \theta(W_y) = \emptyset$.

On the other hand, using Corollary 2.4, we obtain that $f^{-1}(W_x) \times g^{-1}(W_y)$ is ρ-semi-open in $X \times Y$. Moreover, $(x, y) \in f^{-1}(W_x) \times g^{-1}(W_y)$ and $(f^{-1}(W_x) \times g^{-1}(W_y)) \cap A = \emptyset$, so we may conclude that $(x, y) \notin \rho - sCl(A)$.

COROLLARY 4.2. *Let* (X, τ) *and* (Y, σ) *be two topological spaces with operators* α *and* β *associated with* τ *and* σ, *respectively. Consider* $\rho : P(X \times Y) \to P(X \times Y)$ *a monotone operator which is compatible with* α and β. *If* $f : X \to Y$ *is an* (α, β)*-irresolute function and* (Y, σ) *is a* β-semi T_2 *space, then the graph of* f

$$G(f) = \{(x, y) \in X \times Y : y = f(x)\},$$

is ρ*-semi-closed in* $X \times Y$.

PROOF. Let us consider the identity function $id_Y : Y \to Y$. Clearly id_Y is a (β, β)-irresolute function. By the hypothesis $f : X \to Y$ is an

(α, β)-irresolute function and (Y, σ) is a β-semi T_2 space. Using Theorem 4.1, it follows that

$$\{(x,y) \in X \times Y : f(x) = id_Y(y)\} = \{(x,y) \in X \times Y : y = f(x)\} = G(f)$$

is ρ-semi-closed in $X \times Y$.

COROLLARY 4.3. *Let (X,τ) and (Y,σ) be two topological spaces with operators α and β associated with τ and σ, respectively. Consider $\rho : P(X \times Y) \to P(X \times Y)$ a monotone operator which is compatible with α and β. If $f : X \to Y$ is an (α,β)-irresolute function, $B \subseteq Y$ is a semi-compact space, $(Y.\sigma)$ is a β-semi T_2 space, and $G(f)$ is ρ-semi-closed in $X \times Y$, then $f^{-1}(B)$ is an α-semi-closed set in X.*

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2.2. ESTRUCTURAS MINIMALES Y PROPIEDADES DE SEPARACIÓN

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MINIMAL STRUCTURES AND SEPARATIONS PROPERTIES

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Abstract: In this work the notion of m-operator for an m-structure m_X on a set X is introduced. Also several separation forms for points of a set X are described and characterized, in a not necessarily topological context. We also study different relationships between these separation properties, and we establish conditions in regards to the operator which determine the equivalence between these separation forms.

AMS Subject Classification: 54A05, 54A20, 54C08, 54D10
Key Words: m-structure, m-operator, separation properties

1. Introduction

After the works of Levine [5] and Kasahara [4], various mathematician turned their attention in introducing and studying diverse classes of sets related to the notion of operator associated to a topology on a set. Each one of these classes of sets is, in turn, used to obtain different separation properties and new forms of continuity. It is as well as they arise, among others: semi-open, pre-open, β-open, α-semi-open, θ-closed, semi-θ-open, (α,β)-semi-open, $\gamma-(\alpha,\beta)$-semi-open and the different axioms or formulated separation properties respectively, in terms of each of these classes of sets. The descrip-

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tion, the properties and also in the study of situations referred previously, used by Maki [6], in abstract form by means of m-structure or minimal structure notions on a set. In this work, we introduce the notion of m-operator on m-structure and we show that they can be also described, studying the separation properties on a set, without necessarily to have a topology on it. We also find, that the obtained results constitute a generalization of many of the classic results and in particular, those obtained by Caldas et al in [1]. The results that are obtained also provide a conceptual frame that summarizes many relative separation forms in generalized sets derived in a topological space via operators existent in the literature.

2. Minimal Structures

In this section, we introduce the m-structure and the m-operator notions. Also, we define some important subsets associated to these concepts.

Definition 2.1. Let X be a nonempty set and let $m_X \subseteq P(X)$, where $P(X)$ denote the set of power of X. We say that m_X is an m-structure (or a minimal structure) on X, if $\emptyset$ and X belong to m_X.

The members of the minimal structure m_X are called m_X-open sets, and the pair (X, m_X) is called an m-space. The complement of an m_X-open set is said to be an m_X-closed set. An m-structure m_X on a nonempty set X, is said to have the property (B) of Maki, if the union of any family of elements of m_X belongs to m_X. Observe that any collection $\emptyset \neq \mathcal{J} \subseteq P(X)$, always is contained in an m-structure that have the property (B), as we know, $m_X(\mathcal{J}) = \{\emptyset, X\} \cup \{\bigcup_{M \in J} M : \emptyset \neq J \subseteq \mathcal{J}\}$. In particular, when $\mathcal{J} = m_X$, we denote by $m'_X = m_X(\mathcal{J})$. Clearly $m_X = m'_X$, if m_X have the property (B) of Maki. Note that if m_X is an m-structure and $Y \subseteq X$, then $\{M \cap Y : M \in m_X\}$ is an m-structure on Y, and is denoted by $m_{X|Y}$, and the pair $(Y, m_{X|Y})$ is called an m-subspace of (X, m_X).

It is important to observe that the m-structure notion, uses in abstract form the properties of many important collections of generalized sets without the necessity of a topological structure, some of them are illustrated in the following situations:

1. Given a topological space (X, τ), the collections: τ, τ_θ, $SO(X)$, $PO(X)$, $\beta(X)$ are m-structures on X, and all satisfy the property (B). Also, the collection of closed sets in X is an m-structure and satisfy the property (B) of Maki, if (X, τ) is an Alexandroff space.

2. If α is an operator associated with the topology τ on X in the sense of Carpintero et al [2] and [3], then the collections Γ_α and α-$SO(X, \tau)$ are m-structures. Γ_α also has the property (B) and α-$SO(X, \tau)$ has the property (B), if α is a monotone operator.

3. If α, β are operators associated with a topology τ on X, the collection $(\alpha, \beta) - SO(X, \tau)$, introduced by Rosas et al in [9], also is an m-structure and satisfy the property (B).

4. If α, β and γ are operators associated with τ on X, the collection $\gamma - (\alpha, \beta)$-$SO(X, \tau)$, defined by Rosas et al in [11], is also an m-structure and satisfy the property (B), when the operator γ is expansive on the clas $(\alpha, \beta) - SO(X, \tau)$.

Definition 2.2. Let m_X be an m-structure on a set $X \neq \emptyset$. An m operator on m_X, is an application $\alpha : P(X) \to P(X)$ that is expansive o m_X (that is, $U \subseteq \alpha(U)$, for all $U \in m_X$).

A particular case of the previous definition is when $m_X = \tau$, in whic the m-operator notion is exactly the notion of operator associated with th topology introduced by Carpintero et al [2]. Also if α is an m-operator o m_X and $Y \subseteq X$, the restriction $\alpha|_{P(Y)}$ given by $\alpha|_{P(Y)} (M \cap Y) = \alpha(M) \cap Y$ for all $M \subseteq X$, is an m-operator on $m_{X|Y}$.

Definition 2.3. Given two m_X-operators $\alpha, \beta : P(X) \to P(X)$ on m_X We say that $\alpha \preceq \beta$ if $\alpha(U) \subseteq \beta(U)$, for all $U \in m_X$.

Note that $\preceq$ defined previously, is an order on the class $\{\alpha : \alpha$ is a m-operator on $m_X\}$.

Definition 2.4. Let $\alpha : P(X) \to P(X)$ be an m-operator on m_X an $A \subseteq X$. A is called an α-m_X-open set, if for each $x \in A$ there exists a m_X-open set U such that $x \in U$ and $\alpha(U) \subseteq A$. The complement of a α-m_X-open set is an α-m_X-closed set.

We denote the collections of all α-m_X-open sets of X by $O(X, m_X, \alpha$ Observe that the collection $O(X, m_X, \alpha)$ is stable under the union of se and if m_X has the property (B), then we obtain that $O(X, m_X, \alpha) \subseteq m_X$

Also, we note that:

1. If $\alpha = i_{P(X)}$ and m_X is any m-structure satisfying the property $(B$ then the α-m_X-open sets are elements of m_X. In particular, if $m_X =$ where τ is a topology on X, and $\alpha = i_{P(X)}$, the α-m_X-open sets are op sets.

2. If $m_X = \tau$, where τ is a topology on X, and α is an operator associat with τ, the α-m_X-open sets are the α-open sets, described in [8].

3. If $m_X = \beta$-$SO(X,\tau)$, α and β operators associated with τ, where α is expansive on the class β-$SO(X,\tau)$, the α-m_X-open sets are the (α,β)-semi-open sets, described in [9].

4. If $m_X = \gamma$-(α,β)-$SO(X,\tau)$, α,β and γ are operators associated with τ, where γ is expansive on the class (α,β)-$SO(X,\tau)$, the α-m_X-open sets are the γ-(α,β)-semi-open sets, described in [11].

Definition 2.5. Let $\alpha : P(X) \to P(X)$ be an m-operator on m_X and $A \subseteq X$. A is called an α-m_X-semi-open set, if there exist an m_X-open set $U \subseteq X$ such that $U \subseteq A \subseteq \alpha(U)$. The complement of an α-m_X-semi-open set, is called an α-m_X-semi-closed set.

We denote by $SO(X, m_X, \alpha)$ the collection of all α-m_X-semi-open sets of X. Observe that $m_X \subseteq SO(X, m_X, \alpha)$. Also, if m_X has the property (B) of Maki, we obtain that:

$$O(X, m_X, \alpha) \subseteq m_X \subseteq SO(X, m_X, \alpha).$$

Note that the α-semi-open sets, introduced in [2] by Carpintero et al, generalize an extense class of sets in terms of which many generalized separation axioms were described. They constitute a particular case of the previous definition, when $m_X = \tau$ and α is an operator associated with a topology τ.

In general the α-m_X-open sets and the α-m_X-semi-open sets are not stable for the union. Nevertheless, for certain m-operators, the class of α-m_X-semi open sets are stable under union of sets, like it is demonstrated in the following lemma.

Lemma 2.1. *Let m_X be an m-structure which satisfy the property (B) of Maki and let $\alpha : P(X) \to P(X)$ be an m-monotone operator on m_X. If $A_i \in SO(X, m_X, \alpha)$ for each $i \in I$, then $\bigcup_{i\in I} A_i \in SO(X, m_X, \alpha)$.*

Proof. Suppose that m_X has the property (B), α is an m-monotone operator and $A_i \in SO(X, m_X, \alpha)$ for each $i \in I$. For each $i \in I$, there exists a set $U_i \in m_X$ such that $U_i \subseteq A_i \subseteq \alpha(U_i)$, in consequence, $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} \alpha(U_i)$. Since α is a monotone operator, then $\bigcup_{i\in I} \alpha(U_i) \subseteq \alpha(\bigcup_{i\in I} U_i)$; and $\bigcup_{i\in I} U_i \in m_X$, because m_X has the property (B). In consequence, $\bigcup_{i\in I} U_i \in m_X$ and $\bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} A_i \subseteq \alpha(\bigcup_{i\in I} U_i)$, therefore $\bigcup_{i\in I} A_i \in SO(X, m_X, \alpha)$. □

Definition 2.6. Let m_X be an m-structure on X and let α be an m-operator on m_X. We define the $\alpha - m_X$-closure and the $\alpha - m_X$-semi-closure of a set A of X, respectively, as follows:

i) $m_X - \mathrm{sCl}_\alpha(A) = \bigcap\{F : A \subseteq F, X \setminus F \in O(X, m_X, \alpha)\}$,
ii) $m_X - \mathrm{SCl}_\alpha(A) = \bigcap\{F : A \subseteq F, X \setminus F \in SO(X, m_X, \alpha)\}$.

Observe that if $\alpha = i_{P(X)}$ and m_X satisfy the property (B), then the above definition is justly the definition of m-closure, described in [1] and [7], that is;

$$m_X - \mathrm{SCl}_\alpha(A) = m_X - \mathrm{Cl}(A),$$

and

$$m_X - \mathrm{sCl}_\alpha(A) = m_X - \mathrm{SCl}_\alpha(A) = m_X - \mathrm{Cl}(A).$$

Also, we can observe that, for any $A \subseteq X$, the $m_X - \mathrm{sCl}_\alpha(A)$ is an $\alpha - m_X$-closed. But the $m_X - \mathrm{SCl}_\alpha(A)$ is not necessarily an $\alpha - m_X$-semi closed set, but according with Lemma 2.1 and the above definition, the $m_X - \mathrm{SCl}_\alpha(A)$ is an $\alpha - m_X$-semi-closed set when m_X has the property (B) and α is an m-monotone operator.

Note that the condition that α is an m-monotone operator, is not an artificial condition, because, we can find many operators $\alpha : P(X) \to P(X)$ that satisfying such conditions.

Lemma 2.2. *Let m_X be an m-structure on X and let α be an m-operator on m_X. For any subsets A and B of X, the following statements hold:*

1. if $A \subseteq B$, then $m_X - sCl_\alpha(A) \subseteq m_X - sCl_\alpha(B)$.

2. $x \in m_X - sCl_\alpha(A)$ if and only if $U \cap A \neq \emptyset$ for all $\alpha - m_X$-open set U such that $x \in U$;

3. A is an $\alpha - m_X$-closed set if and only if $A = m_X - sCl_\alpha(A)$;

4. $m_X - sCl_\alpha(m_X - sCl_\alpha(A)) = m_X - sCl_\alpha(A)$;

Lemma 2.3. *Let m_X be an m-structure on X and let α be an m-operator on m_X. For any subsets A, B of X, the following statements hold:*

1. If $A \subseteq B$, then $m_X - SCl_\alpha(A) \subseteq m_X - SCl_\alpha(B)$.

2. $x \in m_X - SCl_\alpha(A)$ if and only if $U \cap A \neq \emptyset$ for all $\alpha - m_X$-semi open set U such that $x \in U$.

3. $m_X - SCl_\alpha(m_X - SCl_\alpha(A)) = m_X - SCl_\alpha(A)$;

Also if m_X satisfies the property (B) and α is monotone, then

4. A is an $\alpha - m_X$-semi-closed set if and only if $A = m_X - SCl_\alpha(A)$;

Definition 2.7. Let m_X be an m-structure on X and let α be an m-operator on m_X. A point $x \in X$, is said to be an $\alpha - m_X$-adherent point of a set $A \subseteq X$ if and only if $\alpha(U) \cap A \neq \emptyset$ for all $U \in m_X$ such that $x \in U$.

The set of all $\alpha - m_X$-adherent points of A is denoted by $m_X - Cl_\alpha(A)$. A set A is called (m_X, α)-closed if $m_X - Cl_\alpha(A) = A$. The complement of an (m_X, α)-closed set is an (m_X, α)-open set.

Lemma 2.4. *Let m_X be an m-structure on X and let α be an m-operator on m_X; then*

$$m_X - SCl_\alpha(A) \subseteq m_X - Cl(A) \subseteq m_X - Cl_\alpha(A) \subseteq m_X - sCl_\alpha(A).$$

From the last result, follows that:

$$\alpha - m_X\text{-closed set} \Rightarrow (m_X, \alpha)\text{-closed set} \Rightarrow m_X - \text{closed set}$$

$$\Rightarrow \alpha - m_X\text{-semi-closed set},$$

or equivalently,

$$\alpha - m_X\text{-open set} \Rightarrow (m_X, \alpha)\text{-open set} \Rightarrow m_X - \text{open set}$$

$$\Rightarrow \alpha - m_X\text{-semi-open set},$$

when m_X satisfying the property (B) and α is a monotone operator.

The following definition, generalize the notions of D-sets introduced by Tong in [12].

Definition 2.8. Let m_X be an m-structure on X. A subset $A \subseteq X$, is called an m_X-Difference set (more precisely an m_X-D-set) if there exist subsets U, V in m_X such that $U \neq X$ and $A = U \setminus V$.

Observe that, any m_X-open set $U \neq X$, is an $m_X - D$−set, because trivially $U = U \setminus \emptyset$.

Definition 2.9. Let m_X be an m-structure on X and let α be an m-operator on m_X. A subset $A \subseteq X$, is called an $\alpha - m_X$-generalized closed set (abbreviated by $\alpha - m_X$-sg-closed) if $m_X - \mathrm{sCl}_\alpha(A) \subseteq U$, whenever $A \subseteq U$ and U is an $\alpha - m_X$-open set.

Definition 2.10. Let m_X be an m-structure on X and let α be an m-operator on m_X. A subset $A \subseteq X$, is said to be an $\alpha - m_X$-semi generalized closed set (abbreviated by $\alpha - m_X$-sg-semi-closed) if $m_X - \mathrm{SCl}_\alpha(A) \subseteq U$, whenever $A \subseteq U$ and U is an $\alpha - m_X$-semi-open set.

The followings theorems, characterize the $\alpha - m_X$-generalized closed sets and the $\alpha - m_X$-semi generalized closed sets.

Theorem 2.1. *Let m_X be an m-structure on X that satisfies the property (B) and let α be an m-monotone operator on m_X. $A \subseteq X$ is an $\alpha - m_X$-sg-semi-closed set if and only if there are not exist $\alpha - m_X$-semi-closed set F such that $F \neq \emptyset$ and $F \subseteq m_X - SCl_\alpha(A) \setminus A$.*

Proof. Suppose that A is an $\alpha - m_X$-sg-semi-closed and let $F \subseteq X$ be an $\alpha - m_X$-semi-closed set such that $F \subseteq m_X - \mathrm{SCl}_\alpha(A) \setminus A$. It follows that, $A \subseteq X \setminus F$ and $X \setminus F$ is an $\alpha - m_X$-semi open set, since A is an $\alpha - m_X$-sg-semi-closed, we have that $m_X - \mathrm{SCl}_\alpha(A) \subseteq X \setminus F$ and $F \subseteq X \setminus m_X - \mathrm{SCl}_\alpha(A)$. It follows that

$$F \subseteq m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus m_X - \mathrm{SCl}_\alpha(A)) = \emptyset,$$

implying that $F = \emptyset$. Reciprocally, if $A \subseteq U$ and U is an $\alpha - m_X$-semi-open set, then $m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus U) \subseteq m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus A) = m_X - \mathrm{SCl}_\alpha(A) \setminus A$. Since $m_X - \mathrm{SCl}_\alpha(A) \setminus A$ does not contain subsets $\alpha - m_X$-semi-closed different from the empty set, we obtain that $m_X - \mathrm{SCl}_\alpha(A) \cap (X \setminus U) = \emptyset$, and this implies that $m_X - \mathrm{SCl}_\alpha(A) \subseteq U$ in consequence A is an $\alpha - m_X$-sg-closed. □

In a similar form, we can prove the following characterization.

Theorem 2.2. *Let m_X be an m-structure on X and let α be an m-operator on m_X. $A \subseteq X$ is an $\alpha - m_X$-sg-closed if and only if there are not exist $\alpha - m_X$-closed set F such that $F \neq \emptyset$ and $F \subseteq m_X - sCl_\alpha(A) \setminus A$.*

3. Separation Properties on m-Structures

In this section, we introduce and study different separation properties on a set X with an m-structure. We also look for the existent relation between the different set defined before.

Definition 3.1. Let m_X be an m-structure on X. We say that:

1. X is an m_X-T_0 if for each pair of distinct points $x, y \in X$, there exists an m_X-open sets U of X, such that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.
2. X is an m_X-T_1 if for each pair of distinct points $x, y \in X$, there exists an m_X-open set of X containing x but not y and an m_X-open set of X containing y but not x.
3. X is an m_X-T_2 if for each pair of distinct points $x, y \in X$, there exist disjoint m_X-open sets U and V such that $x \in U$ and $y \in V$.

We can see that the collections $O(X, m_X, \alpha)$ and $SO(X, m_X, \alpha)$ are m-structures in the sense of the Definition 2.1. If we take m_X as $O(X, m_X, \alpha)$ (respectively $SO(X, m_X, \alpha)$), in the Definition 3.1, we obtain separation properties denoted by (m_X, α)-sT_i (respectively (m_X, α)-ST_i), for $i = 0, 1, 2$. From Definition 3.1, it is immediate that:

$$m_X - T_i \Rightarrow m_X - T_{i-1}; \quad (m_X, \alpha) - sT_i \Rightarrow (m_X, \alpha) - sT_{i-1} \quad \text{and}$$

$$(m_X, \alpha) - ST_i \Rightarrow (m_X, \alpha) - ST_{i-1}, \text{ for } i = 1, 2.$$

Definition 3.2. Let m_X be an m-structure on X and let α be an m-operator on m_X. We say that:

1. X is an (m_X, α)-T_0 if for each pair of distinct points $x, y \in X$, there exists an m_X-open set U of X, such that $x \in U$ and $y \notin \alpha(U)$, or $y \in U$ and $x \notin \alpha(U)$.
2. X is an (m_X, α)-T_1 if for each pair of distinct points $x, y \in X$, there exist m_X-open sets U and V of X containing x and y, respectively, such that $y \notin \alpha(U)$ and $x \notin \alpha(V)$.
3. X is an (m_X, α)-T_2 if for each pair of distinct points $x, y \in X$, there exist m_X-open sets U and V of X, such that $x \in U$, $y \in V$ and $\alpha(U) \cap \alpha(V) = \emptyset$.

From Definition 3.2, follows that,

$$(m_X, \alpha) - T_i \Rightarrow (m_X, \alpha) - T_{i-1}, \quad i = 1, 2.$$

Also

$$(m_X, \alpha) - sT_i \Rightarrow (m_X, \alpha) - T_i \Rightarrow m_X - T_i \Rightarrow (m_X, \alpha) - ST_i,$$

for $i = 0, 1, 2$.

It is important to observe that the above definition generalize many of the well known separation axioms seen in the literature. As we specify.

1. For $\alpha = i_{P(X)}$ and m_X any m-structure, the properties of the (m_X, α)-T_i are the separation properties described in the Definitions 3.1, for $i = 0, 1, 2$. 2. Let m_X be any m-structure, ρ an m-operator on m_X and $\alpha = m_X Cl_\rho$. The properties of the (m_X, α)-T_i are the m-Uryshon axioms introduced in [7].

3. For $m_X = \tau$, α an operator associated with τ, the notions of separations described in the above definition are the α-T_i notions introduced in [8].

4. If $m_X = (\alpha, \beta) - SO(X, \tau)$ and the m-operator α is taken as the operator γ, then the above definition is just the separation axioms $\gamma - (\alpha, \beta)$-T_i introduced in [11].

Now we characterize some properties of the m-spaces described above.

Theorem 3.1. *Let m_X be an m-structure on X and let α be an m-operator on m_X. X is an (m_X, α)-ST_0 if and only if for any pair of distinct points x, $y \in X$, we have that $m_X - SCl_\alpha(\{x\}) \neq m_X - SCl_\alpha(\{y\})$.*

Proof. Suppose that X is an (m_X, α)-ST_0, then for any pair of distinct points x, y of X there exists an $\alpha - m_X$-semi-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. It follows from Lemma 2.3, that $m_X - \mathrm{SCl}_\alpha(\{x\}) \neq m_X - \mathrm{SCl}_\alpha(\{y\})$. Reciprocally, if $m_X - \mathrm{SCl}_\alpha(\{x\}) \neq m_X - \mathrm{SCl}_\alpha(\{y\})$, there exists a point $z \in m_X - \mathrm{SCl}_\alpha(\{x\})$ and $z \notin m_X - \mathrm{SCl}_\alpha(\{y\})$, but this implies that, there exists an $\alpha - m_X$-semi-open set U_z such that $x \in U_z$ and $y \notin U_z$. Therefore, X is an (m_X, α)-ST_0. □

Theorem 3.2. *Let m_X be an m-structure on X that satisfies the property (B) and let α be an m-monotone operator on m_X. The following properties are equivalent:*

1. *X is an (m_X, α)-ST_1.*
2. *For any $x \in X$, the unitary set $\{x\}$ is an $\alpha - m_X$-semi-closed set.*
3. *Each subset A of X is the intersection of all $\alpha - m_X$-semi-open sets of X containing A.*

Proof. (1)⇒(2). If X is an (m_X, α)-ST_1 and $x \in X$, then for each $y \in X \setminus \{x\}$, there exists an $\alpha - m_X$-semi-open set U_y such that $y \in U_y$ and $x \notin U_y$, follows that $U_y \cap \{x\} = \emptyset$, therefore $y \in U_y \subseteq X \setminus \{x\}$. In consequence $X \setminus \{x\}$ is an $\alpha - m_X$-semi-open, but this implies that $\{x\}$ is an $\alpha - m_X$-semi-closed set.

(2)⇒(3). Observe that for any $A \subseteq X$, $A = \bigcap_{x \notin A} X \setminus \{x\}$. By hypothesis each unitary set $\{x\}$ is an $\alpha - m_X$-semi-closed set, then each set $X \setminus \{x\}$ with $x \notin A$, is an $\alpha - m_X$-semi-open set.

(3)⇒(1). By hypothesis each unitary set $\{x\}$ is the intersection of all $\alpha - m_X$-semi open sets containing $\{x\}$. In consequence, for each $y \neq x$, there exists an $\alpha - m_X$-semi-open set containing x but not y, follows that X is an (m_X, α)-ST_1. □

In a similar form as the classical case, but not in a necessarily topological context, we have the following separation forms.

Definition 3.3. Let m_X be an m-structure on X and let α be an m-operator on m_X. X is said to be an (m_X, α)-$T_{1/2}$ if each $\alpha - m_X$-sg-closed is an $\alpha - m_X$-semi-closed set.

The following theorem, characterizes the m-spaces satisfying the property (m_X, α)-$ST_{1/2}$.

Theorem 3.3. *Let m_X be an m-structure on X that satisfies the property (B) and let α be an m-monotone operator on m_X. Then X is an (m_X, α)-$ST_{1/2}$ if and only if each unitary set $\{x\}$ in X is an $\alpha - m_X$-semi-open set or an $\alpha - m_X$-semi-closed set.*

Proof. Sufficiency. Suppose that X is an (m_X, α)-$ST_{1/2}$. Then for any $x \in X$, the unitary set $\{x\}$ can be α - m_X-semi-closed set or not. In the case that $\{x\}$ is an $\alpha - m_X$-semi-closed set, the result follows. In the other case, $X \setminus \{x\}$ is an $\alpha - m_X$-sg-closed in m_X. Now using hypothesis, we obtain that $X \setminus \{x\}$ is an $\alpha - m_X$-semi-closed set and therefore, $\{x\}$ is an $\alpha - m_X$-semi-open.

Necessity. Let A be an $\alpha - m_X$-sg-closed in m_X and $x \in m_X - \mathrm{SCl}_\alpha(A)$. If $\{x\}$ is an $\alpha - m_X$-semi-open set, then $\{x\} \cap A \neq \emptyset$ and therefore, $x \in A$. In the case that $\{x\}$ is an $\alpha - m_X$-semi-closed set, then we have that $x \in A$, because if $x \notin A$, then $\{x\} \subseteq m_X - \mathrm{sCl}_\alpha(A) \setminus A$. but this is impossible by Theorem 2.1. □

As a consequence of the last theorem, we have the following corollary.

Corollary 3.1. *Let m_X be an m-structure on X that satisfy the property (B) and let α be an m-monotone operator on m_X. Then X is an (m_X, α)-$ST_{1/2}$ if and only if each subset A of X is the intersection of $\alpha - m_X$-semi-open set and $\alpha - m_X$-semi-closed set that contain A.*

Proof. Sufficiency. Suppose that X is an (m_X, α)-$ST_{1/2}$, since each subset $A \subseteq X$ can be written as $A = \bigcap_{x \notin A} X \setminus \{x\}$. Using Theorem 3.3, we obtain that each $A \subseteq X$ is the intersection of sets that are $\alpha - m_X$-semi-open set or $\alpha - m_X$-semi-closed set that contain A.

Necessity. For each $x \in X$, the set $X \setminus \{x\}$ can be written as the intersections of $\alpha - m_X$-semi-open set and $\alpha - m_X$-semi-closed set that contain $X \setminus \{x\}$, then $\{x\} = \bigcup_{i \in I} S_i$. Here each $S_i \subseteq \{x\}$ and S_i is $\alpha - m_X$-semi-open set or $\alpha - m_X$-semi-closed set. In consequence, for some $j \in I$, we obtain that $\{x\} = S_j$. It follows that $\{x\}$ is an $\alpha - m_X$-semi-open set or an $\alpha - m_X$-semi-closed set . □

Corollary 3.2. *Under the hypothesis of Theorem 3.3, The following statements hold:*

1. $(m_X, \alpha) - ST_{1/2} \Rightarrow (m_X, \alpha) - ST_0$.
2. $(m_X, \alpha) - ST_1 \Rightarrow (m_X, \alpha) - ST_{1/2}$.

In analogous form, under flexible conditions on the m-structure and the m-operator α, we characterize the m-spaces that satisfying the property (m_X, α)-$sT_{1/2}$, as follows.

Theorem 3.4. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then X is an (m_X, α)-$sT_{1/2}$ if and only if each unitary set $\{x\}$ in X is an $\alpha - m_X$-open set or an $\alpha - m_X$-closed set.*

Corollary 3.3. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then X is an (m_X, α)-$sT_{1/2}$ if and only if each subset A of X is the intersection of $\alpha - m_X$-open set and $\alpha - m_X$-closed set that contain A.*

Definition 3.4. Let m_X be an m-structure on X. We say that:

1. X is an m_X-D_0 if for each pair of distinct points $x, y \in X$, there exists an m_X-D-set W of X, such that $x \in W$ and $y \notin W$, or $y \in W$ and $x \notin W$.

2. X is an m_X-D_1 if for each pair of distinct points $x, y \in X$, there exist m_X-D-sets, W and Z in X containing x and y, respectively, such that $y \notin W$ and $x \notin Z$.

3. X is an m_X-D_2 if for each pair of distinct points $x, y \in X$, there exist m_X-D-sets, W and Z of X, such that $x \in W$, $y \in Z$ and $W \cap Z = \emptyset$.

If we take m_X as $O(X, m, \alpha)$ (respectively $SO(X, m, \alpha)$), in the above definition, we obtain the following separation properties, denoted by $(m_X, \alpha)-sD_i$ (respectively $(m_X, \alpha) - SD_i$) for $i = 0, 1, 2$. It is immediate that:

$$m_X - D_i \Rightarrow m_X - D_{i-1}; \quad (m_X, \alpha) - sD_i \Rightarrow (m_X, \alpha) - sD_{i-1},$$

$$(m_X, \alpha) - SD_i \Rightarrow (m_X, \alpha) - SD_{i-1}, \text{ for } i = 1, 2.$$

Also,

$$m_X - T_i \Rightarrow m_X - D_i, \text{ for } i = 0, 1, 2.$$

Theorem 3.5. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then:*

(i) X is an $m_X - D_0$ if and only if X is an $m_X - T_0$.

Also, if m_X satisfies the property (B),

(ii) X is an $m_X - D_1$ if and only if X is an $m_X - D_2$.

Proof. (i) *Necessity.* If X is an $m_X - D_0$ and $x \neq y$, there exist U, $V \in m_X$, $U \neq X$, such that: $x \in U \setminus V$ and $y \notin U \setminus V$ or $y \in U \setminus V$ and $x \notin U \setminus V$. If the case is $x \in U \setminus V$ and $y \notin U \setminus V$, then $x \in U$ and $x \notin V$. Since $y \notin U \setminus V$, can happen that $y \notin U$ or $y \in U$ and $y \in V$. If the case is $y \notin U$ then $x \in U$ and $y \notin U$. In the case that $y \in U$ and $y \in V$, we have that $y \in V$ and $x \notin V$. A similar result is obtaining if $y \in U \setminus V$ and $x \notin U \setminus V$. From the above, we conclude that X is an $m_X - T_0$.

Sufficiency. It is immediate, because all m_X-open set different from X, are m_X-D-sets.

(ii) *Necessity.* Suppose that X is an $m_X - D_1$ and $x \neq y$, then there exist $m_X - D-$ sets $U \setminus V$ and $U' \setminus V'$ such that: $x \in U \setminus V$, $y \notin U \setminus V$, $x \notin U' \setminus V'$ and $y \in U' \setminus V'$. Since $x \notin U' \setminus V'$ then, it can happen some of the following cases: $x \notin U'$ or $x \in U' \cap V'$. If $x \notin U'$, since $y \notin U \setminus V$, we have that: $x \notin U'$, $y \notin U$ or $x \notin U'$, $y \in U \cap V$. In the first case, $x \in U \setminus (U' \cup V)$, because $x \in U \setminus V$, and $y \in U' \setminus (U \cup V')$, but $y \in U' \setminus V'$, also $(U \setminus (U' \cup V)) \cap (U' \setminus (U \cup V')) = \emptyset$. In the second case, we have $x \in U \setminus V$, $y \in V$ and $(U \setminus V) \cap V = \emptyset$. Finally if $x \in U' \cap V'$, then $y \in U' \setminus V'$, $x \in V'$ and $(U' \setminus V') \cap V' = \emptyset$. Therefore in any case, x and y, can be separated by disjoint $m_X - D-$sets, that is, X is an $m_X - D_2$.

Sufficiency. It is immediate. □

Theorem 3.6. *Let m_X be an m-structure on X and let $\alpha : P(X) \to P(X)$ be an application such that $\alpha(U) \cap V = U \cap \alpha(V) = \emptyset$, for any pair of m_X-open sets U and V, $U \cap V = \emptyset$. Then:*

$$m_X\text{-}T_2 \quad \Rightarrow \quad (m_X, \alpha)\text{-}sT_1 \quad \Rightarrow (m_X, \alpha)\text{-}sT_0$$

Proof. Suppose that X satisfies the property m_X-T_2, and x, y two distinct points in X. It follows that, for each point $z \in X \setminus \{y\}$, there exist m_X sets U_z and U_y, such that $z \in U_z$, $y \in U_y$ and $U_z \cap U_y = \emptyset$. Using the property of α, we have that $\alpha(U_z) \cap U_y = \emptyset$, and we obtain the following inclusions, $\alpha(U_z) \subseteq X \setminus U_y \subseteq X \setminus \{y\}$, but this implies that $X \setminus \{y\}$ is an $\alpha - m_X$-open set that contain x but not y. Proceeding in a similar form, we conclude that $X \setminus \{x\}$ is an $\alpha - m_X$-open set that contains y, but not x, we conclude that X is an (m_X, α)-sT_1. Now, using Theorem 3.4, it follows that (m_X, α)-sT_1 implies (m_X, α)-sT_0. □

We can observe, as follows, that there are many situations under which the hypothesis of Theorem 3.6 are satisfied.

1. All the generalized forms of closure (Definition 2.5 and Definition 2.6) on an m-structure are m-operators (in the sense of Definition 2.2) satisfying the conditions of Theorem 3.6.

2. Let $\emptyset \neq Y \subseteq X$ and define an m structure $m_X = \{A \subseteq X : A \cap Y = \emptyset\} \cup X$, the m-operator $\alpha(A) = A \cup Y$, α is an m-operator that satisfies the hypothesis of Theorem 3.6 and $\alpha \neq m_X$-Cl, because $\alpha(\emptyset) = Y$ and $m_X - \mathrm{Cl}\,(\emptyset) = \emptyset$.

3. Clearly if α satisfies the hypothesis of Theorem 3.6, any operator β with $\beta \preceq \alpha$ also satisfies it. Even more, if α and β satisfying the conditions of Theorem 3.6, then $\rho(A) = \alpha(A) \cup \beta(A)$ and $\rho(A) = \alpha(A) \cap \beta(A)$ are also m-operators satisfying the conditions of Theorem 3.6.

In general, the property (m_X, α)-sT_0 does not imply the property m_X-T_2. But, under certain conditions on the application $\alpha : P(X) \to P(X)$ that acts on an m-structure on X, the reverse implication is valid, as we can see in the following theorem.

Theorem 3.7. *Let m_X be an m-structure on X that satisfies the pro-perty B and let $\alpha : P(X) \to P(X)$ be an application that satisfies the following condition $m_X Cl \preceq \alpha$, then: (m_X, α)-sT_0 implies m_X-T_2.*

Proof. Suppose that X is an (m_X, α)-sT_0. For each pair of points x, y in X, such that $x \neq y$, can happen the following cases:

$$\text{(a)}\ x \in U,\, y \notin U\,, \text{ or } \ \text{(b)}\ y \in U,\, x \notin U;$$

for some $\alpha - m_X$-open set U in X.

In the case (a), there exists a set $U_x \in m_X$ such that $x \in U_x$ and $\alpha(U_x) \subseteq U$. By hypothesis $m_X Cl \prec \alpha$, we have that $x \in U_x \subseteq m_X\mathrm{Cl}\,(U_x) \subseteq \alpha(U_x) \subseteq U$. Follows that, $m_X\mathrm{Cl}\,(U_x) \subseteq U$, and since $y \notin U$ we obtain that $y \in X \setminus U \subseteq X \setminus m_X\mathrm{Cl}\,(U_x)$. Therefore, there exist m_X-open sets, U_x and $X \setminus m_X\mathrm{Cl}\,(U_x)$ containing x and y, respectively, such that $U_x \cap (X \setminus m_X\mathrm{Cl}\,(U_x)) = \emptyset$.

In a similar form, we can prove the case (b), that is, there exists an $U_y \in m_X$ such that $y \in U_y$, $x \in X \setminus m_x\mathrm{Cl}\,(U_y)$ also $U_y \cap (X \setminus m_x\mathrm{Cl}\,(U_y)) = \emptyset$, and we conclude that X is an m_X-T_2. □

Observe that Theorem 4.8 in [1], corresponds to the trivial case, because $m_X Cl \preceq m_X Cl$.

An immediate consequence of the last two theorems and Theorem 3.3, is the following corollary.

Corollary 3.4. *Under the hypothesis of Theorems 3.4 and 3.5. The following properties are equivalent:*

1. (X,m) is $(m_X,\alpha)-sD_2$; 2. (X,m) is $(m_X,\alpha)-sD_1$;
3. (X,m) is $(m_X,\alpha)-sD_0$; 4. (X,m) is (m_X,α)-sT_0;
5. (X,m) is (m_X,α)-sT_1; 6. (X,m) is m_X-T_2.

Observe that from the comments on Theorem 3.9, there are many m-operators different from $m_X - Cl$ and $m_X - \mathrm{SCl}_\alpha$ satisfying the hypothesis of Theorems 3.4 and 3.5.

Definition 3.5. Let m_X be an m-structure on X and let $\alpha : P(X) \to P(X)$ be an m-operator on m_X. We say that α is regular respect to m_X, if for each $x \in X$ and each $U \in m_X$ such that $x \in U$, there exists $V \in m_X$ such that $x \in V$ and $\alpha(V) \subseteq U$.

The following theorems characterizes the operators that are regular with respect to an m-structure m_X.

Theorem 3.8. *Let m_X be an m-structure on X and let α be an m-operator on m_X. Then:*

$$\alpha \text{ is regular with respect to an } m_X \iff m_X = \{A : A \text{ is } (m_X,\alpha)\text{-open}\}.$$

Proof. Sufficiency. Suppose that α is regular with respect to m_X and that there exists an m_X-closed subset F such that $m_X - Cl_\alpha(F) \not\subseteq F$. It follows that, there exists a point x such that $x \in m_X - Cl_\alpha(F)$ and $x \notin F$; but this imply that $x \in X \setminus F$, but $X \setminus F$ is an m_X-open set. Now using the hypothesis, there exists $V \in m_X$ such that $x \in V$ and $\alpha(V) \subseteq X \setminus F$, therefore $\alpha(V) \cap F \subseteq (X \setminus F) \cap F = \emptyset$, but this is impossible, because $x \in m_X - Cl_\alpha(F)$. In consequence, all m_X-closed set is an $(m_X,\alpha)^*$-closed, now using Lemma 2.4, it follows that all $(m_X,\alpha)^*$-closed set are m_X-closed.

Necessity. If $m_X = \{A : A$ is an (m_X,α)-open$\}$ and $x \in X$ with $x \in U$, where $U \in m_X$, then, we have that $x \notin X \setminus U = m_X - \mathrm{Cl}(X \setminus U)$, therefore, there exists $V \in m_X$ for which $x \in V$ and $\alpha(V) \cap (X \setminus U) = \emptyset$, it follows that $\alpha(V) \subseteq U$. □

Observe that the above theorem generalizes the characterizations of the regular spaces, in the case when is using the semi regular topology given in [7].

Theorem 3.9. *Let m_X be an m-structure on X and let $\alpha : P(X) \to P(X)$ be an m-operator on m_X. If α is regular with respect to m_X. Then the following properties hold:*

1. All m_X-open set are α-m_X-open set.
2. $m_X - Cl_\alpha(A) = m_X - Cl(A)$, for all $A \subseteq X$.
3. For all m-operator $\beta : P(X) \to P(X)$ on m_X such that $\beta \preceq \alpha$, have:

$$\{A : A \text{ is an } (m_X,\beta)\text{-open set}\} = \{A : A \text{ is } (m_X,\alpha)\text{-open set}\} = m_X.$$

Proof. 1. If $U \in m_X$, then for all $x \in U$ there exists $V_x \in m_X$ such tl $x \in V_x$ and $\alpha(V_x) \subseteq U$. This implies that U is an $\alpha - m_X$-open set.

2. Suppose that $x \notin m_X - \mathrm{Cl}(A)$, it follows that $x \in X \setminus m_X - \mathrm{Cl}($ and $X \setminus m_X - \mathrm{Cl}(A) \in m_X$, but by hypothesis, there exists $V \in m_X$ sı that $x \in V$ and $\alpha(V) \subseteq X \setminus m_X - \mathrm{Cl}(A)$, in consequence, $\alpha(V) \cap A$ $(X \setminus m_X - \mathrm{Cl}(A)) \cap A = \emptyset$, but it implies that $x \notin m_X - Cl_\alpha(A)$. Therefc $m_X - Cl_\alpha(A) \subseteq m_X - \mathrm{Cl}(A)$. Now using Lemma 2.4, we obtain the ot inclusion.

3. If α is a regular operator with respect to m_X and $\beta \preceq \alpha$, it follows t β is regular with respect to m_X and by Lemma 3.1 the result follows.

We can observe, from the last result, that many separation axioms separation properties described before are satisfied for a regular m-opera on m_X.

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2.3. CONJUNTOS m_X-CERRADOS GENERALIZADOS

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Conjuntos $\mathrm{m_X}$-Cerrados Generalizados

m_X-Generalized Closed Sets

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Resumen

Dado (X, m_X) un m-espacio, se introduce el concepto de conjunto m_X-g-cerrado como una generalización de las definiciones de varias clases de conjuntos cerrados generalizados. Se obtiene un nuevo axioma de separación, denominado $m_X - T_{1/2}$ y se caracterizan éstos. También se estudian las relaciones entre los m-espacio $m_X - T_{1/2}$ y los m-espacio $m_X - T_0$ y $m_X - T_1$.
Palabras Claves: m_X-estructura, conjunto cerrado generalizado, m-espacio $m_X - T_{\frac{1}{2}}$.

Abstract

Given (X, m_X) an m-space, we introduce the concept of m_X-g-closed set as a generalization of the definitions of several classes of generalized closed sets. Also we obtain and characterize a new separaton axiom called $m_X - T_{1/2}$. Also we study the relations between the m-space $m_X - T_{1/2}$ and the m-spaces $m_X - T_0$ and $m_X - T_1$.
Key words and phrases: m_X-structure, generalized closed set, m-space $m_X - T_{\frac{1}{2}}$.

1 Introducción

Los conjuntos cerrados, semi-cerrados, α-semi-cerrados y (α, β)-semi-cerrados han sido utilizados por varios autores para definir diferentes clases de conjuntos cerrados generalizados y con estos introducir nuevos axiomas de separación.

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En 1963, Levine [5] introduce el concepto de conjuntos g-cerrados y en 1991 Ogata [7] define los espacios $T_{1/2}$, también introduce las nociones de conjuntos s-g-cerrados y espacios semi-$T_{1/2}$. En el 2000 Rosas, Carpintero, Vielma y Salas [12] estudian el concepto de conjuntos α-sg-cerrado y caracterizan los espacios α-semi-$T_{1/2}$. En el 2005 Rosas, Carpintero y Sanabria [11] definen los conjuntos (α, β)-sg-cerrados y estudian los espacios (α, β)-semi-$T_{1/2}$.

En este artículo utilizamos la noción de estructura minimal m_X sobre un conjunto no vacío X dada por $Maki$ [6] y definimos los conjuntos m_X-g-cerrados como una generalización de los conjuntos g-cerrados, sg-cerrado, α-g-cerrado, α-sg-cerrado y (α, β)-sg-cerrado. También se definen y se caracterizan los $m_X - T_{1/2}$ que genera
lizan, de forma natural, a los espacios $T_{1/2}$, $\alpha - T_{1/2}$, semi-$T_{1/2}$, α-semi-$T_{1/2}$ y (α, β)-semi-$T_{1/2}$.

2 Preliminares

Sea X un conjunto no vacío, se dice que $\alpha : \mathcal{P}(X) \to \mathcal{P}(X)$ es un *operador expansivo* sobre una familia Γ de subconjuntos de X si $U \subset \alpha(U)$ para todo $U \in \Gamma$. Si (X, τ) es un espacio topológico y α es un operador expansivo sobre la topología τ, entonces diremos que α es un *operador asociado* a la topología τ [3]. Además, si $\alpha(A) \subset \alpha(B)$ siempre que $A \subset B$, entonces decimos que el operador α es monótono.

Si (X, τ) es un espacio topológico, α un operador expansivo sobre la topología τ y A es un subconjunto de X, entonces ese dice que A es α-abierto [1] si para cada $x \in A$ existe un abierto U de x tal que $\alpha(U) \subset A$. El complemento de un conjunto α-abierto se denomina α-cerrado, se define la α-clausura de un subconjunto A de X, abreviada por $\alpha - cl(A)$, como la intersección de todos conjuntos α-cerrados que contienen a A. Se prueba que la $\alpha - cl(A)$ es un conjunto α-cerrado. Un conjunto A es α-cerrado generalizado, abreviado por α-g-cerrado, si $\alpha - cl(A) \subset U$ siempre que $A \subset U$ y U es α-abierto. Todo conjunto α-cerrado es α-g-cerrado. Se definen los espacios $\alpha - T_{\frac{1}{2}}$ como aquellos espacios en los cuales los conjuntos α-g-cerrado y α-cerrado coinciden, de modo que X es $\alpha - T_{\frac{1}{2}}$ sí y solo si para todo $x \in X$ se tiene que $\{x\}$ es α-cerrado o α-abierto. La colección de todos los subconjuntos α-abierto de X se denota por τ_α

Un subconjunto A de X es α-semi-abierto [12] si existe un conjunto abierto $U \in \tau$ tal que $U \subset A \subset \alpha(U)$. El complemento de un conjunto α-semi-abierto se denomina α-semi-cerrado. Se define la α-semi-clausura de A, abreviado por $\alpha - scl(A)$, como la intersección de todos los conjuntos α-semi-cerrados que contienen a A; si α es un operador monótono entonces $\alpha - scl(A)$ es un conjunto α-semi cerrado. Se dice que A es un conjunto α-semi-cerrado generalizado, abreviado por α-sg-cerrado, si $\alpha - scl(A) \subset U$ siempre que $A \subset U$ y U es un conjunto α-semi-abierto. Si α es monótono, todo conjunto α-semi-cerrado es un conjunto α-sg-cerrado. Se dice que X es un espacio $\alpha - semiT_{\frac{1}{2}}$ si todo conjunto α-sg-cerrado es α-semi-cerrado, de modo que X es $\alpha - semiT_{\frac{1}{2}}$ sí y solo si para todo $x \in X$ se tiene que $\{x\}$ es α-semi-cerrado o α-semi-abierto. La colección de todos los subconjuntos α-semi-abierto de X se denota por $\alpha - SO(X)$. Si β es otro operador asociado a τ, entonces un subconjunto A de X es (α, β)-semi-abierto [11] si para cada $x \in A$ existe un conjunto β-semi-abierto V tal que $x \in V$ y $\alpha(V) \subset A$. El complemento de un conjunto (α, β)-semi-abierto se denomina (α, β)-semi-cerrado. Se define la (α, β)-semi-clausura de A, abreviada por $(\alpha, \beta) - scl(A)$, como la intersección de todos los conjuntos (α, β)-semi-cerrados que contienen a A; se prueba que $(\alpha, \beta) - scl(A)$ es un conjunto (α, β)-semi-cerrado. Un subconjunto A de X es un conjunto (α, β)-semi-cerrado generalizado, abreviado (α, β)-sg-cerrado, si $(\alpha, \beta) - scl(A) \subset U$ siempre que $A \subset U$ y U es un conjunto (α, β)-semi-abierto. Todo conjunto (α, β)-semi-cerrado es un conjunto (α, β)-sg-cerrado. Se dice que X es un espacio $(\alpha, \beta) - semiT_{\frac{1}{2}}$ si todo conjunto (α, β)-sg-cerrado es (α, β)-semi-cerrado, de modo que X es $(\alpha, \beta) - semiT_{\frac{1}{2}}$ sí y solo si para todo $x \in X$ se tiene que $\{x\}$ es (α, β)-semi-cerrado o (α, β)-semi-abierto. La colección de todos los subconjuntos (α, β)-semi-abierto de X se denota por $(\alpha, \beta) - SO(X)$

3 Estructuras Minimales

En esta sección se plantea el concepto de estructura minimal [6] y algunas de sus propiedades. También se define la noción de m-espacios$m_X - T_1$ y se caracterizan en funci'on de sus conjuntos unitarios.

Definición 3.1. [6] Una estructura minimal o una m_X-estructura sobre un conjunto no vacío X, es una familia m_X de subconjuntos de X tal que $\emptyset \in m_X$ y $X \in m_X$.

El par (X, m_X) formado por un conjunto no vacío X y una m_X estructura sobre X, se denomina m-espacio. Cada elemento de m_X se denomina conjunto m_X-abierto y el complemento de un conjunto m_X-abierto se denomina conjunto m_X-cerrado. Si (X, τ) es un espacio topológico, α y β son operadores asociados a la topología entonces las colecciones τ, τ_α, $\alpha - SO(X)$ y $(\alpha, \beta) - SO(X)$ son m_X-estructuras.

Definición 3.2 ([6])**.** Sean (X, m_X) un m-espacio y A un subconjunto de X, se define la m_X clausura de A, abreviada $m_X - cl(A)$, como la intersección de todos los conjuntos m_X-cerrados que contienen a A, es decir

$$m_X - cl(A) = \bigcap \{F : F \supseteq A, X \setminus F \in m_X\}$$

Observe que si $X \setminus A \in m_X$, entonces $m_X - cl(A) = A$, es decir, si A es m_X-cerrado, entonces $m_X - cl(A) = A$; además $A \subset m_X - cl(A)$. Otras propiedades de $m_X - cl(A)$, se enuncian en el siguiente teorema.

Teorema 3.1 ([6])**.** *Sea (X, m_X) un m-espacio, A y B subconjuntos de X. Las siguientes se satisfacen.*

1. $m_X - cl(\emptyset) = \emptyset$.
2. $m_X - cl(X) = X$.
3. *Si $A \subset B$, entonces $m_X - cl(A) \subset m_X - cl(B)$.*
4. $m_X - cl(A \cup B) \supseteq m_X - cl(A) \cup m_X - cl(B)$.
5. $m_X - cl\,(m_X - cl(A)) = m_X - cl(A)$.

Teorema 3.2 ([8])**.** *Sea (X, m_X) un m-espacio, A un subconjunto de X y $x \in X$, entonces $x \in m_X - cl(A)$ si y sólo si $U \cap A \neq \emptyset$ para todo $U \in m_X$ tal que $x \in U$.*

Definición 3.3 ([6])**.** Si m_X es una estructura minimal sobre X tal que la unión de elementos de m_X es un elemento de m_X, entonces diremos que m_X satisface la condición (B) de Maki.

Observe que si m_X satisface la condición (B) de Maki, entonces la intersección de conjuntos m_X-cerrados es un conjunto m_X-cerrado y por tanto, si $A \subset X$, entonces $m_X - cl(A)$ es un conjunto m_X-cerrado.

Teorema 3.3. *[6] Sea (X, m_X) un m-espacio y A un subconjunto de X. Si m_X satisface la condición (B) de Maki entonces, A es m_X-cerrado sí y sólo si $m_X - cl(A) = A$.*

En el teorema anterior, si la condición (B) de Maki es removida, es posible tener un m-espacio (X, m_X) y un subconjunto A de X para el cual $m_X - cl(A) = A$ y A no sea un conjunto m_X-cerrado, como se observa en el siguiente ejemplo.

Ejemplo 1. Considere $X = \{a, b, c, d\}$ con la siguiente m_X estructura,

$$m_X = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}.$$

Sea $A = \{c, d\}$. Observe que $m_X - cl(A) = A$ y A no es un conjunto m_X-cerrado.

Definición 3.4 ([10])**.** Sea (X, m_X) un m-espacio, se dice que X es $m_X - T_0$ si para cada par de puntos distintos $x, y \in X$, existe $U \in m_X$ tal que $x \in U$ y $y \notin U$ o $x \notin U$ y $y \in U$.

Definición 3.5 ([9])**.** Se (X, m_X) un m-espacio, se dice que X es $m_X - T_2$ si para cada par de puntos distintos $x, y \in X$, existen conjuntos disjuntos $U, V \in m_X$ que contienen a x e y respectivamente.

Definición 3.6. Sea (X, m_X) un m-espacio, se dice que X es $m_X - T_1$ si para cada par de puntos distintos $x, y \in X$ existen conjuntos $U, V \in m_X$ que contienen a x e y respectivamente y satisfacen que $y \notin U$ y $x \notin V$.

Observe que

$$m_X - T_2 \Rightarrow m_X - T_1 \Rightarrow m_X - T_0.$$

Los siguientes ejemplos muestran que existen m-espacios que son $m_X - T_0$ pero no $m_X - T_1$, y $m_X - T_1$ que no son $m_X - T_2$

Ejemplo 2. Considere el conjunto de los números reales $\mathbb{R}$ con la estructura minimal

$$m_{\mathbb{R}} = \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R} \setminus \{x\} : x \in \mathbb{R}\}.$$

$\mathbb{R}$ es $m_{\mathbb{R}} - T_1$, pues dado $x, y \in \mathbb{R}$ con $x \neq y$, podemos encontrar $m_{\mathbb{R}}$-abiertos $U = \mathbb{R} \setminus \{y\}$ y $V = \mathbb{R} \setminus \{x\}$ que contienen a x e y respectivamente y $x \notin V$, $y \notin U$.

Supongamos que $\mathbb{R}$ es $m_{\mathbb{R}} - T_2$, es decir, para $x, y \in \mathbb{R}$ con $x \neq y$ existen conjuntos $U, V \in m_{\mathbb{R}}$ disjuntos tales que $x \in U$ e $y \in V$. Entonces $U = \mathbb{R} \setminus \{a_1\}$ y $V = \mathbb{R} \setminus \{a_2\}$ con $a_1 \neq x$ y $a_2 \neq y$. Esto significa que $U \cap V = \mathbb{R} \setminus \{a_1, a_2\} \neq \emptyset$ lo que implica que $\mathbb{R}$ no es $m_{\mathbb{R}} - T_2$.

Ejemplo 3. Consideremos $\mathbb{R}$ con la siguiente $m_{\mathbb{R}}$ estructura

$$m_{\mathbb{R}} = \{\emptyset, \mathbb{R}\} \cup \{[x, \infty) : x \in \mathbb{R}\}.$$

$\mathbb{R}$ es $m_{\mathbb{R}} - T_0$ pues para $x, y \in \mathbb{R}$ con $x < y$, $U = [y, \infty)$ es $m_{\mathbb{R}}$-abierto y $x \notin U$ e $y \in U$.

$\mathbb{R}$ no es $m_{\mathbb{R}} - T_1$ pues cualquier $m_{\mathbb{R}}$-abierto U que contenga a x es de la forma $U = [a, \infty)$ con $a \leq x$ y por tanto $y \in U$.

Los teoremas que se enuncian a continuación caracterizan las nociones de $m_X - T_0$ y $m_X - T_1$.

Teorema 3.4. *Sea (X, m_X) un m-espacio, X es $m_X - T_0$ si y sólo si para todo par de puntos distintos $x, y \in X$ se cumple que $m_X - cl\,(\{x\}) \neq m_X - cl\,(\{y\})$.*

Demostración. Supongamos que X es $m_X - T_0$ y sean $x, y \in X$ tales que $x \neq y$, entonces existe $U \in m_X$ tal que $x \in U$ y $y \notin U$ o $y \in U$ y $x \notin U$. Sin perdida de generalidad, podemos suponer que existe $U \in m_X$ tal que $x \in U$ y $y \notin U$. Entonces si $m_X - cl\,(\{x\}) = m_X - cl\,(\{y\})$, se tiene $x \in m_X - cl\,(\{y\})$ y por tanto, $U \cap \{y\} \neq \emptyset$, en contradicción que $y \notin U$. Así se debe tener $m_X - cl\,(\{x\}) \neq m_X - cl\,(\{y\})$.

Recíprocamente, sean $x, y \in X$ tales que $x \neq y$ y supongamos que $m_X - cl\,(\{x\}) \neq m_X - cl\,(\{y\})$, entonces existe $z \in X$ tal que $z \in m_X - cl\,(\{x\})$ y $z \notin m_X - cl\,(\{y\})$ o viceversa.

Sin perdida de generalidad, podemos suponer que $z \in m_X - cl\,(\{x\})$ y $z \notin m_X - cl\,(\{y\})$, entonces existe $V \in m_X$ tal que $z \in V$ y $V \cap \{y\} = \emptyset$ y $V \cap \{x\} \neq \emptyset$, es decir $y \notin V$ y $x \in V$, es decir, X es $m_X - T_0$. $\square$

Teorema 3.5. *Sea (X, m_X) un m-espacio. Si para cada $x \in X$ se tiene que $\{x\}$ es m_X-cerrado, entonces X es $m_X - T_1$. El recíproco es cierto si m_X satisface la condición (B) de Maki.*

Demostración. Sean $x, y \in X$ tales que $x \neq y$, entonces $\{x\}$ y $\{y\}$ son conjuntos m_X-cerrados y por lo tanto $X \setminus \{y\}$ y $X \setminus \{x\}$ son conjuntos m_X-abiertos que contienen a x e y respectivamente y se cumple que $y \notin X \setminus \{y\}$ y $x \notin X \setminus \{x\}$, de donde se concluye que X es $m_X - T_1$.

Recíprocamente, supongamos que X es $m_X - T_1$ y que m_X satisface la condición (B) de Maki. Sea $x \in X$, entonces para cada $y \in X$ con $x \neq y$ existen conjuntos $U, V \in m_X$ que contienen a x e y respectivamente y satisfacen que $x \notin V$ y $y \notin U$, es decir, $\{x\} \cap V = \emptyset$. Por lo que $y \notin m_X - cl(\{x\})$. Por lo tanto, $m_X - cl(\{x\}) = \{x\}$ y como m_X satisface la condición (B) de Maki, entonces $\{x\}$ es m_X-cerrado. $\square$

Observe que la condición (B) de Maki es suficiente para caracterizar los m-espacios que son $m_X - T_1$. El siguiente ejemplo nos muestra que el recíproco del teorema anterior, en general no es cierto si no se le exige la condición (B) de Maki a la m_X estructura.

Ejemplo 4. Consideremos $\mathbb{R}$ con la siguiente $m_\mathbb{R}$ estructura

$$m_\mathbb{R} = \{\emptyset, \mathbb{R}\} \cup \{\{x\} : x \in \mathbb{R}\}.$$

$\mathbb{R}$ es $m_\mathbb{R} - T_1$ pues para $x \neq y$, $\{x\}$, $\{y\}$ son $m_\mathbb{R}$-abiertos que no contienen a y y x respectivamente. Sin embargo $\{x\}$ no es $m_\mathbb{R}$-cerrado para ningún $x \in \mathbb{R}$.

4 Conjuntos m_X-g-Cerrados y $m_X - T_{\frac{1}{2}}$

En esta sección, utilizando la noción de m_X estructura, se generalizan los conceptos de conjunto g-cerrado [5], α-g-cerrado [12],α-sg-cerrado [12], (α,β)-sg-cerrado [11] y de espacios $T_{\frac{1}{2}}$ [5], $\alpha - T_{\frac{1}{2}}$ [12], $\alpha - semi - T_{\frac{1}{2}}$ [12], y $(\alpha,\beta) - semi - T_{\frac{1}{2}}$ [11].

Definición 4.1. Sea (X, m_X) un m-espacio, A un subconjunto de X, se dice que A es un conjunto m_X-cerrado generalizado, abreviado m_X-g-cerrado, si $m_X - cl(A) \subset U$ siempre que $A \subset U$ y $U \in m_X$.

Si (X,τ) es un espacio topológico y α y β son operadores asociados a la topología, entonces esta definición coincide con los conceptos de conjuntos g-cerrados [5], α-g-cerrado[12], (α,β)-sg-cerrado[11], cuando la m_X estructura es la colección τ,τ_α, $(\alpha,\beta) - SO(X)$ respectivamente. Además si m_X satisface la condición (B) de Maki, entonces este concepto coincide con la noción de conjunto α-sg-cerrado [12], cuando la m_X estructura es la colección $\alpha - SO(X)$.

Teorema 4.1. *Sea (X, m_X) un m-espacio, todo conjunto m_X-cerrado es m_X-g-cerrado.*

Los siguientes ejemplos nos muestran la existencia de conjuntos en un m-espacios que son m_X-g-cerrado que no son m_X-cerrado.

Ejemplo 5. Consideremos $\mathbb{R}$ con la siguiente $m_\mathbb{R}$ estructura

$$m_\mathbb{R} = \{\emptyset, \mathbb{R}\} \cup \{\{x\} : x \in \mathbb{R}\}.$$

El conjunto $\mathbb{Q}$ de los números racionales es $m_\mathbb{R}$-g-cerrado pues el único $m_\mathbb{R}$-abierto que lo contiene es $\mathbb{R}$; pero no es $m_\mathbb{R}$-cerrado.

Ejemplo 6. Consideremos $\mathbb{R}$ con la siguiente $m_{\mathbb{R}}$ estructura

$$m_{\mathbb{R}} = \{\emptyset, \mathbb{R}\} \cup \{[x, \infty) : x \in \mathbb{R}\}.$$

el conjunto $\mathbb{Q}$ de los números racionales es un conjunto $m_{\mathbb{R}}$-g-cerrado pues el único $m_{\mathbb{R}}$-abierto que lo contiene es $\mathbb{R}$; pero no es $m_{\mathbb{R}}$-cerrado.

Definición 4.2. Sea (X, m_X) un m-espacio, se dice que X es $m_X - T\frac{1}{2}$, si todo conjunto m_X-g-cerrado es m_X-cerrado.

Es de observar que la condición (B) de Maki no necesariamente la satisfacen los m-espacios que son $m_X - T_0$ o $m_X - T_1$ o $m_X - T_2$, sin embargo existe una estrecha relación entre los m-espacios que son $m_X - T\frac{1}{2}$ y los m-espacios que satisfacen la condición (B) de Maki, como se observa en el siguiente teorema.

Teorema 4.2. *Sea (X, m_X) un m-espacio, si X es $m_X - T\frac{1}{2}$ entonces m_X satisface la condición (B) de Maki.*

Demostración. Supongamos que m_X no satisface la condición (B) de Maki, entonces existe una colección $\{U_\alpha\}_{\alpha \in J}$ de conjuntos m_X abiertos tal que $\cup_{\alpha \in J} U_\alpha \notin m_X$, luego $F = X \setminus \cup_{\alpha \in J} U_\alpha$ no es un conjunto m_X cerrado. Veamos que F es un conjunto m_X-g-cerrado.

En efecto, sea $V \in m_X$ tal que $F \subset V$, entonces

$$\begin{aligned} m_X - cl(F) &= m_X - cl(X \setminus \cup_{\alpha \in J} U_\alpha) \\ &= m_X - cl(\cap_{\alpha \in J}(X \setminus U_\alpha)) \\ &\subseteq \cap_{\alpha \in J} m_X - cl(X \setminus U_\alpha) \\ &= \cap_{\alpha \in J}(X \setminus U_\alpha) \\ &= X \setminus \cup_{\alpha \in J} U_\alpha = F \subset V \end{aligned}$$

De modo que F es un conjunto m_X-g-cerrado que no es m_X-cerrado y por tanto X no es $m_X - T\frac{1}{2}$. □

Teorema 4.3. *Sea (X, m_X) un m espacio y A un subconjunto de X. Si A es m_X-g-cerrado entonces $m_X - cl(A) \setminus A$ no contiene subconjuntos m_X-cerrados no vacíos. El recíproco es cierto si m_X satisface la condición (B) de Maki.*

Demostración. Supongamos que A es un conjunto m_X-g-cerrado y sea K un subconjunto m_X-cerrado de $m_X - cl(A) \setminus A$, entonces $X \setminus K$ es un conjunto m_X-abierto que contiene a A y por tanto $m_X - cl(A) \subset X \setminus K$, de modo que $K \subset (X \setminus m_X - cl(A)) \cap (m_X - cl(A))$, de donde se concluye que $K = \emptyset$

Recíprocamente, supongamos que $m_X - cl(A) \setminus A$ no contiene subconjuntos m_X-cerrados no vacíos y que m_X satisface la condición (B) de Maki. Sea $U \in m_X$ tal que $A \subset U$, entonces $m_X - cl(A) \cap (X \setminus U)$ es un conjunto m_X-cerrado y

$$m_X - cl(A) \cap (X \setminus U) \subset m_X - cl(A) \cap (X \setminus A) = m_X - cl(A) \setminus A$$

por tanto $m_X - cl(A) \cap (X \setminus U) = \emptyset$, es decir $m_X - cl(A) \subset U$ de donde se concluye que A es m_X-g-cerrado. □

El siguiente teorema caracteriza los m-espacios $m_X - T\frac{1}{2}$.

Teorema 4.4. *Sea (X, m_X) un m-espacio, X es $m_X - T\frac{1}{2}$ sí y sólo si las siguientes se satisfacen:*

1. *Para todo $x \in X$ se tiene que $\{x\}$ es m_X-abierto o m_X-cerrado.*

2. *La m_X estructura satisface la condición (B) de Maki.*

Demostración. Sea $x \in X$ y supongamos que $\{x\}$ no es m_X-cerrado, entonces $X \setminus \{x\}$ no es m_X-abierto, de modo que el único m_X-abierto que contiene a $X \setminus \{x\}$ es X, por lo que $X \setminus \{x\}$ es trivialmente $m_X - g$-cerrado, por hipótesis X es $m_X - T_{1/2}$, entonces $X \setminus \{x\}$ es m_X-cerrado y por tanto $\{x\}$ es m_X-abierto. Por Teorema 4.2 se concluye que m_X satisface la condición (B) de Maki.

Recíprocamente, sea A un conjunto m_X-g-cerrado y $x \in m_X - cl(A)$, entonces por hipótesis puede ocurrir

1. $\{x\}$ sea m_X-abierto, entonces $\{x\} \cap A \neq \emptyset$ y por tanto $x \in A$; es decir, $m_X - cl(A) \subset A$.

2. $\{x\}$ se m_X-cerrado, como A es m_X-g-cerrado, entonces $m_X - cl(A) \setminus A$ no contiene conjuntos m_X-cerrados no vacíos, entonces $x \in A$; es decir, $m_X - cl(A) \subset A$.

En cualquier caso $m_X - cl(A) = A$; como m_X satisface la condición (B) de Maki, entonces A es m_X-cerrado. □

Existen m-espacios en los cuales los conjuntos unitarios son m_X-abiertos o m_X-cerrados y sin embargo el m-espacios X no es $m_X - T\frac{1}{2}$, tal y como se muestra en el siguiente ejemplo.

Ejemplo 7. Considere $X = \{a, b, c, d\}$ con la siguiente m_X estructura,

$$m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b, d\}, \{a, b, c\}\}$$

Observe que los conjuntos unitarios son m_X-abiertos o m_X cerrados y sin embargo X no es $m_X - T\frac{1}{2}$. En efecto, $\{c, d\}$ es un conjunto m_X-g-cerrado pues el único m_X-abierto que lo contiene es X, y $\{c, d\}$ no es un conjunto m_X-cerrado.

Los siguientes teoremas muestran la relación existente entre los m-espacios $m_X - T_0$, $m_X - T_1$ y $m_X - T\frac{1}{2}$.

Teorema 4.5. *Sea (X, m_X) un m-espacio, si X es $m_X - T_{1/2}$ entonces X es $m_X - T_0$.*

Demostración. Sean $x, y \in X$ con $x \neq y$, entonces $\{x\}$ es m_X-abierto o m_X-cerrado.

Si $\{x\}$ es m_X-abierto entonces para $V = \{x\}$ se tiene que $x \in V$ y $y \notin V$.

Si $\{x\}$ es m_X-cerrado, entonces $V = X \setminus \{x\}$ es m_X-abierto y $x \notin V$ y $y \in V$. Por tanto X es $m_X - T_0$. □

A continuación se exhibe un m-espacio que es $m_X - T_0$ pero que no es $m_X - T\frac{1}{2}$.

Ejemplo 8. Consideremos $\mathbb{R}$ con la siguiente $m_{\mathbb{R}}$ estructura

$$m_{\mathbb{R}} = \{\emptyset, \mathbb{R}\} \cup \{[x, \infty) : x \in \mathbb{R}\}.$$

$\mathbb{R}$ es $m_{\mathbb{R}} - T_0$; sin embargo no es $m_{\mathbb{R}} - T_{1/2}$.

Teorema 4.6. *Sea (X, m_X) un m-espacio, si m_X satisface la condición (B) de Maki y X es $m_X - T_1$, entonces X es $m_X - T_{1/2}$.*

Demostración. Si X es $m_X - T_1$ y satisface la condición (B) de Maki, entonces para todo $x \in X$ se tiene que $\{x\}$ son m_X-cerrados y por tanto X es $m_X - T_{1/2}$. □

El siguiente ejemplo nos muestra un m-espacio que es $m_X - T_{1/2}$ pero que no es $m_X - T_1$.

Ejemplo 9. Consideremos $\mathbb{R}$ con la siguiente $m_{\mathbb{R}}$ estructura

$$m_{\mathbb{R}} = \{\emptyset, \mathbb{R}, \{a\}\} \cup \{\mathbb{R} \setminus \{x\} : x \neq a\},$$

donde a es un número real fijo. $m_{\mathbb{R}}$ satisface la condición (B) de Maki y los unitarios son conjuntos $m_{\mathbb{R}}$-abiertos o $m_{\mathbb{R}}$-cerrado. Por tanto $\mathbb{R}$ es $m_{\mathbb{R}} - T_{1/2}$; pero no es $m_{\mathbb{R}} - T_1$ porque $\{a\}$ no es $m_{\mathbb{R}}$-cerrado.

Es de observar que existen m-espacios que son $m - T_1$ pero que no son $m - T_{1/2}$. Para ello es suficiente encontrar un m-espacio que sea $m - T_1$ pero que no satisfaga la condición (B) de Maki

Ejemplo 10. Consideremos $\mathbb{R}$ con la siguiente $m_{\mathbb{R}}$ estructura

$$m_{\mathbb{R}} = \{\emptyset, \mathbb{R}\} \cup \{\{x\} : x \in \mathbb{R}\}.$$

$\mathbb{R}$ es $m_{\mathbb{R}} - T_1$; pero no es $m_{\mathbb{R}} - T_{1/2}$. En efecto, $\mathbb{Q}$ es $m_{\mathbb{R}}$-g-cerrado pues el único $m_{\mathbb{R}}$-abierto que lo contiene es $\mathbb{R}$; pero no es $m_{\mathbb{R}}$-cerrado.

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2.4. FAMILIAS INADMISIBLES Y PRODUCTO DE TOPOLOGÍAS GENERALIZADAS

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Inadmissible Families and Product of Generalized Topologies

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1. Introduction

Császár introduce in [6],[7],[8] and [9] the concept of generalized topology and associated notions. Later, he introduces the notion of generalized topology on a Cartesian product of sets and obtained several properties of it [9]). Carpintero, Rosas and Sanabria introduced ([4]) a new class of associated operators on the product topology in which each factor of the product space has an associated operator to the respective topology. In this work, we characterize the finitely inadmissible collections of subsets in the Cartesian product. Also, we study the Cartesian product of γ-compact and the Cartesian product of γ-semi compact spaces according to [9] and [4]. As a consequence, we obtain a general framework which allows us to derive in a unified way many results about of generalized compactness in the Cartesian product of generalized topologies.

2. Preliminary

In this section, we recall some concepts and basic results defined by Császár in [6],[7],[8] and [9].

Let X be a nonempty set and we denote by $\exp X$ its power set. A collection $\mu \subseteq \exp X$ of subsets of X is said to be a *generalized topology* on X (briefly a GT on X) if $\emptyset \in \mu$ and an arbitrary union of elements of μ belongs to μ (In general $X \notin \mu$. If $X \in \mu$, μ is said to be a *strong* GT on X). The elements of μ are said to be *μ-pen set*, their complements *μ-closed sets* .

Let $\mathcal{B} \subseteq \exp X$ satisfy $\emptyset \in \mathcal{B}$. Then all unions of some elements of $\mathcal{B}$ constitute a GT $\mu(\mathcal{B})$, and $\mathcal{B}$ is said to be a *base* for $\mu(\mathcal{B})$.

Given a GT μ on X, Császár ([6], [7]) define mappings i_μ, $c_\mu : \exp(X) \rightarrow \exp(X)$ as follows:

$$\begin{aligned} i_\mu A &= \quad \{B \in \mu : B \subseteq A\}, \\ c_\mu A &= \quad \{C : X - C \in \mu, C \supseteq A\}. \end{aligned}$$

Observe that $i_\mu A$ is the largest μ-open subset of A, and $c_\mu A$ is the smallest μ-closed subset of X containing A. In the following lemmas several properties of i_μ, $c_\mu : \exp(X) \rightarrow \exp(X)$ are considered (see [6, Lemmas 1.1, 1.4]).

Lemma 2.1. *The operation $i_\mu : exp(X) \rightarrow \exp(X)$ fulfils:*

(i) *$A \subseteq B \subseteq X$ implies $i_\mu A \subseteq i_\mu B$;*
(ii) *$i_\mu A \subseteq A$;*
(iii) *$i_\mu i_\mu A = i_\mu A$.*

Lemma 2.2. *For $c_\mu : exp(X) \rightarrow \exp(X)$, we have:*

(i) *$A \subseteq B \subseteq X$ implies $c_\mu A \subseteq c_\mu B$;*
(ii) *$c_\mu A \subseteq A$;*
(iii) *$c_\mu c_\mu A = c_\mu A$.*

Also, we have the followings characterizations.

Lemma 2.3. *Let μ be a GT on a set X. Then:*

(i) *$x \in i_\mu A$ if and only if there exists $M \in \mu$ such that $x \in M \subseteq A$;*
(ii) *$x \in c_\mu A$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$*

In [7], Császár considered several collections of $\exp X$ for a given GT μ on X.

$$
\begin{array}{llll}
(1) & \sigma(\mu) & = & \{A \in \exp X : A \subseteq c_\mu i_\mu A\}, \\
(2) & \pi(\mu) & = & \{A \in \exp X : A \subseteq i_\mu c_\mu A\}, \\
(3) & \alpha(\mu) & = & \{A \in \exp X : A \subseteq i_\mu c_\mu i_\mu A\}, \\
(4) & \beta(\mu) & = & \{A \in \exp X : A \subseteq c_\mu i_\mu c_\mu A\}, \\
(5) & \zeta(\mu) & = & \{A \in \exp X : A \subseteq c_\mu i_\mu A \cup i_\mu c_\mu A\}.
\end{array}
$$

When μ is a topology on X, the elements of $\sigma(\mu)$, (respectively $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$, $\zeta(\mu)$) are said to be *semi-open*, (respectively *preopen*, *β-open*, *b-open*).

According Császár ([9]), if $K \neq \emptyset$ is an index set, $X_k \neq \emptyset$ for $k \in K$, and $X = \Pi_{k\in K} X_k$ the Cartesian product of the set X_k. Suppose that, for $k \in K$, μ_k is a given GT on X_k. Let us consider all sets of the form $\Pi_{k\in K} M_k$ where $M_k \in \mu_k$ and, with the exception of a finite number of indices k, $M_k = M_{\mu_k}$. We denote by $\mathcal{B}$ the collection of all these sets, and we define a GT $\mu = \mu(\mathcal{B})$ having $\mathcal{B}$ for *base*. We call μ the *product* of the GT's μ_k and denote it by $P_{k\in K}\mu_k$.

Remark 2.4. *Let p_k be the k^{th}* projection *$p_k : X \to X_k$. For a given k and $M_k \in \mu_k$, we denote $< M_k >= p_k^{-1}(M_k)$; this is the "slab" in $X = \Pi_{k\in K} X_k$ where each factor is X_k except the k^{th}, which is M_k. Similarly, for finitely many indices $k_1, \dots k_n$ in K and sets $M_{k_1} \in \mu_{k_1}, \dots M_{k_n} \in \mu_{k_n}$, the subset*

$$< M_{k_1} > \cap \dots \cap < M_{k_n} >= p_{k_1}^{-1}(M_{k_1}) \cap \dots \cap p_{k_n}^{-1}(M_{k_n}) = \mathop{}\limits_{i=1}^{n} p_{k_i}^{-1}(M_{k_i}),$$

is denoted by $< M_{k_1} \dots M_{k_n} >$ (see [11]). So, all unions of subsets of the form $< M_{k_1} \dots M_{k_n} >$ above defined constitute $\mathcal{B}$ for $\mu = P_{k\in K}\mu_k$. Moreover, if $x \in M \in P_{k\in K}\mu_k$ there exists a subset $< M_{k_1} \dots M_{k_n} >$, such that $x \in< M_{k_1} \dots M_{k_n} >\subseteq M$. Equivalently, if $M \in P_{k\in K}\mu_k$ then $M = \ _{i\in I} G_i$ (I being a non empty set of indices), where $G_i = \ _{r\in J(i)} p_r^{-1}(M_r^i)$, $J(i)$ being a finite subset of K for all $i \in I$, and M_r^i being a γ_r-open set in the r^{th} factor X_r, for any $r \in J(i)$ and $i \in I$.

We shall refer to [9] for more details and results concerning the product of generalized topologies.

3. γ-open and γ-semi open sets in GT's

In this section, we consider the notion of operator (or operation) on a set, and the notions of γ-open and γ-sets in generalized topology. Also, we introduce a new class of operation associated with the product of GT's, in order to obtain extension of the results given by Carpintero, Rosas and Sanabria (see [5]) in the framework of GT's.

Let X be a nonempty set, and $\gamma : \exp X \to \exp X$ a mapping. We call $\gamma : \exp X \to \exp X$ an *operation* on X ([5],[9]) if it is monotone (i.e $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$). If both γ and γ' are operations, then by composition we can obtain the operation $\gamma \circ \gamma'$ (or simply $\gamma\gamma'$ instead $\gamma \circ \gamma'$).

Given a GT μ on a set X, we can obtain important special cases of operations when γ is taken as follows: $\gamma = c_\mu$, $\gamma = i_\mu$, and its compositions $\gamma = c_\mu i_\mu$, $i_\mu c_\mu$, $i_\mu c_\mu i_\mu$, $c_\mu i_\mu c_\mu$.

Considering an operation γ on the set, Császár [6], [8] introduced the notion of γ-open sets. A subset $A \subseteq X$ is said to be γ-open set, if $A \subseteq \gamma(A)$. A large literature is devoted to γ-open sets if μ is a GT on X and γ is taking as $c_\mu i_\mu$, (respectively $i_\mu c_\mu$,$i_\mu c_\mu i_\mu$,$c_\mu i_\mu c_\mu$ or $c_\mu i_\mu(A) \cup i_\mu c_\mu(A)$), the collection of the corresponding γ-open sets is $\sigma(\mu)$, (respectively $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$, $\zeta(\mu)$).

The following lemma shows the behavior of γ-open sets under the union of sets.

Lemma 3.1. [6]*Let X be a nonempty set and μ be a GT on X. If $\gamma : \exp X \to \exp X$ is an operation and $\{A_i : i \in I\}$ is a collection of γ-open sets in X, then the union $\bigcup_{i\in I} A_i$ is a γ-open set in X.*

In a natural way, we can introduce the γ-semi open sets.

Definition 3.2. *Let X be a nonempty set, μ be a GT on X, and $\gamma : \exp X \to \exp X$ be an operation. A subset $A \subseteq X$ is said to be γ-semi open set, if $M \subseteq A \subseteq \gamma(A)$, for some $M \in \mu$.*

Now, we introduce the following generalization of the notion of associated operator to a topology in the GT's.

Definition 3.3. *Let X be a nonempty set and μ be a GT on X. An operation $\gamma : \exp X \to \exp X$ is said to be associated to μ if $M \subseteq \gamma(M)$, for all $M \in \mu$.*

Observe that for an operation $\gamma : \exp X \to \exp X$, each γ-semi open set is a γ-open set. In fact, A γ-semi open implies that, there exists an μ-open set M such that $M \subseteq A \subseteq \gamma(M)$. Since γ is monotone, $A \subseteq \gamma(M) \subseteq \gamma(A)$. In general,
γ-open does not implies that γ-semi open. However, we have the following equivalence.

The following example show that γ-open set does not γ-semi open set.

Example 3.4. *Let $\{a,\ b,\ c,\ d\}$. Define a GT as follows $\mu = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Let γ defined as $\gamma(M) = c_\mu M$. Then, we have $\{b,c\}$ is γ-*
open set but not γ-semi open set.

Theorem 3.5. *Let X be a nonempty set, μ be a GT on X and $\gamma : \exp X \to \exp X$ be an operation. Then, $A \subseteq X$ is an γ-semi open set if and only if A is an γ'-open set, where $\gamma' := \gamma i_\mu$.*

Proof. Necessity. Suppose that A is an γ-semi open set. Then, we have $M \subseteq A \subseteq \gamma(M)$ such that $M \in \mu$. Since γ is monotone, we obtain $A \subseteq \gamma((i_\mu(A))$. Thus, A is an $\gamma' = \gamma i_\mu$-open set.

Sufficiency. This is immediate consequence of Definition 3.2, since $i_\mu(A))$ is μ-
open set.

From Theorem 3.5 and Lemma 3.1, we have Lemma 3.6 as follows:

Lemma 3.6. *Let X be a nonempty set and let μ a GT on X. If $\gamma : \exp X \to \exp X$ be an operation and $\{A_i : i \in I\}$ is a collection of γ-semi open sets in X, then the union $\bigcup_{i\in I} A_i$ is a γ-semi open set in X.*

Remark 3.7. *By the Lemma 3.1 the collection of γ-open sets is an GT on X. Similarly, by the Lemma 3.5, the collection of γ-semi open sets is an GT on X.*

We now define a class of operations on the product of generalized topologies.

Definition 3.8. *Let $K \neq \emptyset$ be an index set. Suppose that, for each $k \in K$ μ_k are given a GT on $X_k \neq \emptyset$ and an operation $\gamma_k : \exp X_k \to \exp X_k$ on X_k. An operation $\gamma : \exp X \to \exp X$ on $X = \prod_{k\in K} X_k$, is said to be compatible with $\{\gamma_k\}_{k\in K}$, if*

$$\gamma(< M_{k_1} \dots M_{k_n} >) =< \gamma_{k_1}(M_{k_1}) \dots \gamma_{k_n}(M_{k_n}) >,$$

for each member $< M_{k_1} \dots M_{k_n} >$ in the base of $\mu = P_{k\in K}\mu_k$.

Let $K \neq \emptyset$ be an index set. Suppose that μ_k is given a GT on $X_k \neq \emptyset$, for each $k \in K$ and considering $\mu = P_{k\in K}\mu_k$ on $X = \prod_{k\in K} X_k$. Let us write $i = i_\mu$, $c = c_\mu$, $i_k = i_{\mu_k}$ and $c_k = c_{\mu_k}$.

In the following example, we shows important cases of operations on the product compatible with the operations on its factors.

Example 3.9. *Let be*

$$\gamma_k = c_k\, (resp., i_k, c_k i_k, i_k c_k, i_k c_k i_k, c_k i_k c_k).$$

for each $k \in K$. Then, acoording to Propositions 2.2 and 2.3 in [9]

$$\gamma = c\, (resp., i, ci, ic, ici, cic),$$

is in each case an operation on X compatible with the γ_k's.

By means of class of operations above defined, we can obtain important relationships between the structure of the γ-semi open (resp,. γ-open) sets in the product and the structure of the γ_k-semi open (resp,. γ_k-open) sets in each of its factors X_k.

Lemma 3.10. *Let $K \neq \emptyset$ be an index set. Suppose that, for each $k \in K$, a GT μ_k on $X_k \neq \emptyset$ and an operation $\gamma_k : \exp X_k \to \exp X_k$ on X_k, associated to μ_k, are given. Suppose that $\gamma : \exp X \to \exp X$ is an operation on $X = \prod_{k\in K} X_k$, associated to $\mu = P_{k\in K}\mu_k$ and compatible with $\{\gamma_k\}_{k\in K}$, such that $\gamma(\emptyset) = \emptyset$. If $\emptyset \neq \prod_{k\in K} A_k$, $A_k \subseteq X_k$, is a γ-semi open set in X, then A_k is γ_k-semi open set in X_k for each $k \in K$.*

Proof. Suppose that $\emptyset \neq \prod_{k\in K} A_k$ is a γ-semi open set in X. Then there exists an μ-open set $M \subseteq X$ such that $M \subseteq \prod_{k\in K} A_k \subseteq \gamma(M)$. It is clear that $M \neq \emptyset$ because if $M = \emptyset$. Then $\emptyset \neq \prod_{k\in K} A_k \subseteq \gamma(\emptyset) = \emptyset$, this is impossible by the hypothesis. Let $p_k : X \to X_k$ be the k^{th} projection. Then $p_k(M) \subseteq p_k(\prod_{k\in K} A_k) = A_k$. Hence $p_k(M) \subseteq A_k$, for each $k \in K$. On the other hand, for all $k \in K$ we have

$$M \subseteq \prod_{k\in K} p_k(M) \subseteq < p_k(M) > .$$

By hypothesis γ is monotone and compatible with $\{\gamma_k\}_{k\in K}$. Since $p_k : X \to X_k$ is (μ, μ_k)-open (see [9, Proposition 2.4]), $p_k(M) \in \mu_k$ for all $k \in K$, so we obtain

$$\prod_{k\in K} A_k \subseteq \gamma(M) \subseteq \gamma(< p_k(M) >) = < \gamma_k(p_k(M)) > .$$

This implies that $A_k \subseteq \gamma_k(p_k(M))$ for each $k \in K$.

By the above argument, we see that there exists an γ_k-open set $p_k(M) \subseteq X_k$ which satisfies

$$p_k(M) \subseteq A_k \subseteq \gamma_k(p_k(M)),$$

from which we conclude that each A_k is an γ_k-semi open set in X_k.

Corollary 3.11. *Under the hypothesis of Lemma 3.10, if the product $\prod_{k\in K} A_k$, is a nonempty proper subset and γ-semi open set of X, then there exists a finite subset $\{k_1, k_2, ..., k_n\} \subseteq K$ such that the γ_k-semi open sets A_k are distint from X_k, for each $k \in \{k_1, k_2, ..., k_n\}$.*

Proof. By hypothesis there exists an μ-open set $\emptyset \neq M \subseteq X$ such that

$$M \subseteq \prod_{k\in K} A_k \subseteq \gamma(M).$$

Consequently there exists a point $x \in M$ and a basic set $< M_{k_1} ... M_{k_n} >$ in $\mu = P_{k\in K}\mu_k$, such that

$$x \in < M_{k_1} ... M_{k_n} > \subseteq M \subseteq \prod_{k\in K} A_k \subseteq \gamma(M).$$

It follows that $x \in < M_{k_1} \dots M_{k_n} > \subseteq \; {}_{k\in K} A_k$. But this implies that $X_k = A_k$, for each $k \notin \{k_1, k_2, ..., k_n\}$.

Theorem 3.12. *Let $K \neq \emptyset$ be an index set. Suppose that a GT μ_k on $X_k \neq \emptyset$ for each $k \in K$ and an operation $\gamma_k : \exp X_k \to \exp X_k$ on X_k, associated to μ_k, are given. Suppose that $\gamma : \exp X \to \exp X$ is an operation on $X = \; {}_{k\in K} X_k$, associated to $\mu = P_{k\in K}\mu_k$ and compatible with $\{\gamma_k\}_{k\in K}$, such that $\gamma(\emptyset) = \emptyset$. Then*

$$< A_{k_1}, A_{k_2}, \dots A_{k_n} > \text{ is } \gamma\text{-semi open} \Leftrightarrow A_{k_i} \text{ is } \gamma_{k_i}\text{-semi open, } i = 1, ..., n.$$

Proof. (Sufficience). It follows from Lemma 3.10.

(Necessity)Let A_{k_i}, $A_{k_i} \neq X_{k_i}$, be a γ_{k_i}-semi open set in X_{k_i} for each $i \in \{1, 2, ..., n\}$. By hypothesis, there exists μ_{k_i}-open sets $M_{k_i} \subseteq X_{k_i}$ such that $M_{k_i} \subseteq A_{k_i} \subseteq \gamma_{k_i}(M_{k_i})$, for $i = 1, ..., n$. Note that from $M_{k_i} \subseteq A_{k_i} \neq X_{k_i}$, we obtain that $M_{k_i} \neq X_{k_i}$ for $i \in \{1, 2, ..., n\}$. Therefore

$$\begin{aligned} < M_{k_1}, M_{k_2}, \dots M_{k_n} > &\subseteq \; < A_{k_1}, A_{k_2}, \dots A_{k_n} > \\ &\subseteq \; < \mu_{k_1}(M_{k_1}), \mu_{k_2}(M_{k_2}), \dots \mu_{k_n}(M_{k_n}) >, \end{aligned}$$

from which we obtain that

$$< M_{k_1}, M_{k_2}, \dots M_{k_n} > \subseteq < A_{k_1}, A_{k_2}, \dots A_{k_n} > \subseteq \gamma(< M_{k_1}, M_{k_2}, \dots M_{k_n} >).$$

Thus $< A_{k_1}, A_{k_2}, \dots A_{k_n} >$ is μ-semi open set.

Remark 3.13. *In the Theorem 3.12, we obtain a generalization of the results proved by Carpintero, Rosas and Sanabria (Lemmas 2.1, 2.2 and Theorem 2.3 in* [4]*). Moreover, the above result implies Proposition 2.7 in* [9]*, proved by Császár under the assumption that every μ_k is strong. Obviously, in the proof of the Theorem 3.12 this hypothesis is unnecessary.*

Theorem 3.14. *Let $K \neq \emptyset$ be an index set. Suppose that, for each $k \in K$, a GT μ_k on $X_k \neq \emptyset$ and an operation $\gamma_k : \exp X_k \to \exp X_k$ on X_k, associated to μ_k, are given. Suppose that $\gamma : \exp X \to \exp X$ is an operation on $X = \; {}_{k\in K} X_k$, associated to $\mu = P_{k\in K}\mu_k$ and compatible with $\{\gamma_k\}_{k\in K}$ such that $\gamma(\emptyset) = \emptyset$, then for all $k \in K$ we have:*

(i) *$p_k^{-1}(M_k)$ is γ-semi open in X, if M_k is γ_k-semi open in X_k;*
(ii) *$p_k(V)$ is γ_k-semi open in X_k, if V is γ-semi open in X.*

Proof.
(i) Follows from the Theorem 3.12.

(ii) Suppose that V is a γ-semi open set in $\; {}_{k\in K} X_i$. Then there exists an γ-open set $M \subseteq \; {}_{k\in K} X_k$ such that $M \subseteq V \subseteq \gamma(M)$. If $M = \emptyset$, then

$$\emptyset = p_k(\emptyset) = p_k(M) \subseteq p_k(V) \subseteq p_k(\gamma(M)) = p_k(\gamma(\emptyset)) = p_k(\emptyset) = \emptyset.$$

So $p_k(V) = \emptyset$, and trivially $p_k(V)$ is γ_k-semi open in X_k. On the other hand, if $M \neq \emptyset$, then $M = \ _{i\in I} G_i$ (I a non empty set of indices) since M is a γ-open set in $\ _{k\in K} X_k$, , where $G_i = \ _{r\in J(i)} p_r^{-1}(M_r^i)$, being $J(i)$ a finite subset of K for all $i \in I$, and M_r^i be a γ_r-open set in the r^{th} factor X_r, for any $r \in J(i)$ and $i \in I$. Now, for each $i \in I$, we have that either

(a) exists $i_0 \in I$ such that $k \notin J(i_0)$, or (b) $k \in J(i)$ for all $i \in I$.

In case (a), we observe that $G_{i_0} \subseteq \ _{i\in I} G_i = M$, then $p_k(G_{i_0}) \subseteq p_k(M)$. But, $p_k(G_{i_0}) = p_k(\ _{r\in J(i_0)} p_r^{-1}(M_r^{i_0})) = X_k$, because $i \neq r$ for all $r \in J(i_0)$. Thus $p_k(G_{i_0}) = X_k$, from this and by inclusion $M \subseteq V$, it then follows that

$$X_k = p_k(\gamma(G_{i_0})) \subseteq p_k(\ _{i\in i} G_i) = p_k(M) \subseteq p_k(V) \subseteq X_k,$$

hence $p_k(V) = X_k$. Therefore $p_k(V)$ is γ_k-semi open in X_k. In case (b), observe that

$$p_k(G_i) = p_k(\ _{r\in J(i)} p_r^{-1}(M_r^k)) = M_r^i \quad , \forall i \in I,$$

since $k \in J(i)$, for all $i \in I$. Then,

$$p_k(M) = p_k(\ _{i\in I} G_i) = \ _{i\in I} p_k(G_i) = \ _{i\in I} M_r^i.$$

Thus, $\ _{i\in I} M_r^i = p_k(M) \subseteq p_k(V)$. Observe that

$$M \subseteq \ _{k\in K} p_k(M) \subseteq < p_k(M) >,$$

M is a γ-open set, p_k is a (μ, μ_k)-open (see [9, Proposition 2.4]) and γ is a monotone associated operation compatible with $\{\gamma_k\}$, we obtain that

$$V \subseteq \gamma(M) \subseteq < \gamma_k(p_k(M)) > = < \gamma_k(p_k(\ _{i\in I} U_r^i))) > .$$

In consequence,

$$p_k(V) \subseteq p_k(< \gamma_k(p_k(\ _{i\in I} M_r^i))) >) = \gamma_k(p_k(\ _{i\in I} M_r^i))).$$

Thus, $\ _{i\in I} U_r^i \subseteq p_k(V) \subseteq \gamma_k(\ _{i\in I} M_r^i)$, hence $p_k(V)$ is a γ_k-semi open set in X_k.

Remark 3.15. *In the Theorem 3.13, we obtain extensions of Proposition 2.4 in* [9] *and Proposition 2.7 in* [9]*, which are particular cases of Theorem 3.12, by taking $\gamma(A) = A$, for all $A \subseteq X$, and $\gamma_k(A_k)$ for $A_k \subseteq X_k$.*

4. Inadmissible Families and Products

In this section we summarized some terminology and results concerning finitely inadmissible families. Also we give characterizations for those finitely inadmissible families in a Cartesian product of sets.

Recall that a collection $\mathcal{A}$ of subsets of a set $X \neq \emptyset$, is said to be an inadmissible (or inadequate) [23] if $\mathcal{A}$ fails to covers X. $\mathcal{A}$ is said to be finitely inadmissible (or finitely inadequate) family, briefly f.i, if no finite subcollection of $\mathcal{A}$ covers X.

The property of being finitely inadmissible satisfies the following conditions.

Lemma 4.1. [15] *Let X be a nonempty set and let $\mathcal{A}$ be a nonempty collection of subsets of X. The following assertions hold:*

(i) *If $\mathcal{A}$ is f.i, then $\mathcal{A}$ has finite character;*
(ii) *If $\mathcal{A}$ is f.i and $A \subseteq X$, then either $\mathcal{A} \cup \{A\}$ or $\mathcal{A} \cup \{X \setminus A\}$ is f.i.*

Using the Tukey's Lemma, or equivalently, the axiom of choice, we obtain the following result.

Lemma 4.2. *Let $\mathcal{A}$ be a f.i family of subsets of X. Then:*

(i) *There exists a f.i family $\mathcal{A}^+$ of subsets of X such that, $\mathcal{A} \subseteq \mathcal{A}^+$ and $\mathcal{A}^+$ is maximal with respect to the partial order*
$$\mathcal{A} \prec \mathcal{A}' \text{ if and only if } \mathcal{A} \subseteq \mathcal{A}',$$
defined on the class of all f.i families of subsets of X containing $\mathcal{A}$.
(ii) *If $A \notin \mathcal{A}^+$ and $A \subseteq A'$, then $A' \notin \mathcal{A}^+$*

Proof.
(i) It is a direct consequence of Tukey's Lemma, or the axiom of choice (see [4],[15]).

(ii) If $A \notin \mathcal{A}^+$, from the maximality of $\mathcal{A}^+$, it follows that $\mathcal{A}^+ \cup \{A\}$ is not f.i. By the Lemma 4.1, either $\mathcal{A}^+ \cup \{A\}$ or $\mathcal{A}^+ \cup \{X \setminus A\}$ is f.i. If $\mathcal{A}^+ \cup \{A\}$ is not f.i, then $\mathcal{A}^+ \cup \{X \setminus A\}$ is f.i. Since $A \subseteq A'$, then $X = A \cup (X \setminus A) \subseteq A' \cup (X \setminus A)$. We have $X = A' \cup (X \setminus A)$. In consequence, if $A' \in \mathcal{A}^+$ then $\mathcal{A}^+ \cup \{X \setminus A\}$ is not f.i, but it is impossible. Thus $A' \notin \mathcal{A}^+$.

Applying this lemma, we obtain the following result.

Lemma 4.3. *Let X be a nonempty set. If $\mathcal{A}'$ and $\mathcal{A}$ are families of subsets of X such that, $\mathcal{A}' \subseteq \mathcal{A}$ and each member of $\mathcal{A}$ is a superset of some member of $\mathcal{A}'$. Then, $\mathcal{A}$ contains a f.i subfamily if and only if $\mathcal{A}'$ contains a f.i subfamily*

Proof.
(Sufficiency). Suppose that $\mathcal{A}$ contains a f.i subfamily $\mathcal{B}$. By hypothesis, for each $B \in \mathcal{B}$, $B \in \mathcal{A}$ and there exists $A' \in \mathcal{A}'$ such that $A' \subseteq B$. If

$\mathcal{B}^+$ is the maximal f.i family corresponding to the collection $\mathcal{B}$, then we have $A' \subseteq B \in \mathcal{B} \subseteq \mathcal{B}^+$ from the Lemma 4.1. Thus, $A' \subseteq B$ and $B \in \mathcal{B}^+$. By part (ii) of the Lemma 3.1, necessarily $A' \in \mathcal{B}^+$. Hence there exists $A' \in \mathcal{B}^+ \cap \mathcal{A}'$, for all $B \in \mathcal{B}$. On the other hand, $\mathcal{B}^+ \cap \mathcal{A}' \subseteq \mathcal{B}^+$ and $\mathcal{B}^+$ is f.i, then $\mathcal{B}^+ \cap \mathcal{A}'$ is f.i. Therefore $\mathcal{B}' = \mathcal{B}^+ \cap \mathcal{A}'$, thus $\mathcal{B}' \subseteq \mathcal{A}'$ and $\mathcal{B}'$ is f.i.

(Necessity). It is obvious since $\mathcal{A}' \subseteq \mathcal{A}$.

Let $K \neq \emptyset$ be an index set. Suppose that a nonempty collection $\mathcal{A}_k$ of subsets of a set X_k for each $k \in K$ is given. Now we consider the following subcollections of Cartesian product ${}_{k\in K} X_k$ of the sets X_k.

i) ${}_{k\in K} \mathcal{A}_k$ denotes the collection of all subset $V \subseteq {}_{k\in K} X_k$, such that there exists $k \in K$ and $U_k \in \mathcal{A}_k$ for which $p_k^{-1}(U_k) \subseteq V$,

ii) ${}_{k\in K} \mathcal{A}_k$ denotes the collection of all subset of Cartesian product ${}_{k\in K} X_k$, whose members are union of sets of the form $p_k^{-1}(U_k)$, where $U_k \in \mathcal{A}_k$ and $k \in K$.

Trivially ${}_{k\in K} \mathcal{A}_k \subseteq {}_{k\in K} \mathcal{A}_k$, but not necessarily ${}_{k\in K} \mathcal{A}_k = {}_{k\in K} \mathcal{A}_k$. For the collections above defined, using the Lemma 4.3, we obtain the following result.

Lemma 4.4. ${}_{k\in K} \mathcal{A}_k$ *(resp.,* ${}_{k\in K} \mathcal{A}_k$*) contains a f.i subfamily if and only if, for some* $k \in K$*,* $\mathcal{A}_k$ *contains a f.i subfamily.*

Proof.

(Sufficiency) Suppose that ${}_{k\in K} \mathcal{A}_k$ contains a f.i family and for each $k \in K$, the collection $\mathcal{A}_k$ does not contain any f.i subfamily. By definition of ${}_{k\in K} \mathcal{A}_k$ and Lemma 4.3, there exists a subcollection $\mathcal{C}$, whose members are of type $p_k^{-1}(U_k)$, with $U_k \in \mathcal{A}_k$ and $k \in K$, such that $\mathcal{C}$ does not contain any finite subcollection that covers ${}_{k\in K} X_k$. For each $k \in K$, consider

$$\mathcal{C}_k = \{U_k : U_k \in \mathcal{A}_k \text{ y } p_k^{-1}(U_k) \in \mathcal{C}\}.$$

Observe that $\mathcal{C}_k \subseteq \mathcal{A}_k$, and by hypothesis $\mathcal{A}_k$ does not contain any f.i subfamily, so there exists a finite subcollection $\{U_{k_1}, U_{k_2}, ..., U_{k_n}\}$ of $\mathcal{C}_k$ such that $X_k = {}_{i=1}^{n} U_{k_i}$. Then,

$$\underset{k\in K}{} X_k = p_k^{-1}(X_k) = p_k^{-1}(\overset{n}{\underset{i=1}{}} U_{k_i}) = \overset{n}{\underset{i=1}{}} p_k^{-1}(U_{i_k}).$$

Hence ${}_{k\in K} X_k = {}_{i=1}^{n} p_k^{-1}(U_{k_i})$, but it is impossible because $p_k^{-1}(U_{k_i}) \in \mathcal{C}$, for all $i = 1, 2, ..., n$ and $k \in K$.

(Necessity) Suppose that any subfamily of ${}_{k\in K} \mathcal{A}_k$ is not f.i. For each $k \in K$ and for all subcollection $\mathcal{B}_k \subseteq \mathcal{A}_k$, trivially, we have that $\{p_k^{-1}(U_k) : U_k \in \mathcal{B}_k\}$ is a subcollection of ${}_{k\in K} \mathcal{A}_k$. By hypothesis, there exists a finite subcollection $\{B_{k_1}, B_{k_2}, ..., B_{k_n}\} \subseteq \mathcal{B}_k$, such that $\ _{k\in K} X_k = \ _{i=1}^{n} p_k^{-1}(B_{k_i})$. From this equality, since p_k is onto, it follows that

$$X_k = p_k(\ _{k\in K} X_k) = p_k(\ _{i=1}^{n} p_k^{-1}(B_{k_i})) = \ _{i=1}^{n} p_k(p_k^{-1}(B_{k_i})) = \ _{i=1}^{n} B_{k_i}.$$

Thus $X_k = \ _{i=1}^{n} B_{k_i}$, this implies that $\mathcal{B}_k$ is not f.i. Therefore $\mathcal{A}_k$ does not contain any f.i subfamily.

In the case of $\ _{k\in K} \mathcal{A}_k$. Observe that $\ _{k\in K} \mathcal{A}_k \subseteq \ _{k\in K} \mathcal{A}_k$, and by Lemma 4.3, $\ _{k\in K} \mathcal{A}_k$ contains a f.i subcollection if and only if $\ _{k\in K} \mathcal{A}_k$ contains a f.i subcollection. But, by the above proof we conclude that there exists an collection $\mathcal{A}_k$ which contains a f.i subfamily.

In the next theorem, we get the following generalization of Lemma 4.4.

Theorem 4.5. *If $\mathcal{A} \subseteq P(\ _{k\in K} X_i)$ and $\mathcal{A}_k \subseteq P(X_k)$, for all $k \in K$, are collections that satisfying the following conditions:*

i) $p_k^{-1}(U_k) \in \mathcal{A}, \quad \forall U \in \mathcal{A}_k, \forall k \in K,$
ii) $p_k(V) \in \mathcal{A}_k, \quad \forall V \in \mathcal{A}, k \in K,$
iii) *$\mathcal{A}$ is stable for the union of sets.*

Then, $\mathcal{A}$ contains a f.i subfamily if and only if $\mathcal{A}_k$ contains a f.i subcollection, for some $k \in K$

Proof.
(Sufficiency) Suppose that $\mathcal{A}$ contains a f.i subfamily $\mathcal{B}$. Then, for any finite subcollection $\{B_1, B_2, ..., B_n\} \subseteq \mathcal{B}$, $\ _{k\in K} X_k \neq \ _{i=1}^{n} B_i$. Thus, there exists an element $x = (x_k)_{k\in K} \in \ _{k\in K} X_k$ such that, $x \notin \ _{i=1}^{n} B_i$. From this, it follows that for all element $y = (y_k)_{k\in K} \in \ _{i=1}^{n} B_i$, $x \neq y$. Hence, there exists $k_0 \in K$ such that $x_{k_0} \neq y_{k_0} = p_{k_0}(y)$. Then $x_{k_0} \neq p_{k_0}(y)$, for all $y \in \ _{i=1}^{n} B_i$, this implies that $x_{k_0} \notin p_{k_0}(\ _{k=1}^{n} B_i) = \ _{i=1}^{n} p_{k_0}(B_i)$. In consequence, $X_{k_0} \neq \ _{i=1}^{n} p_{k_0}(B_i)$, from which we conclude, by hypothesis ii), that $\{p_{k_0}(B_1), p_{k_0}(B_2), ..., p_{k_0}(B_n)\} \subseteq \mathcal{A}_{k_0}$ is f.i.

(Necessity) Suppose that $\mathcal{A}_k$, $k \in K$, contains a f.i subcollection. By Lemma 4.4, we have that $\ _{k\in K} \mathcal{A}_k$ contains a f.i subcollection. From the hypothesis i) and iii), it follows that $\ _{k\in K} \mathcal{A}_k \subseteq \mathcal{B}$, so $\mathcal{B}$ contains a f.i subfamily.

Observe that if $\mathcal{A}$ contains a subcollection $\mathcal{B}$, such that $\mathcal{B}$ is non-inadmissible and $\mathcal{B}$ is f.i. Following the proof of previous theorem, it easy to see that for some $k_0 \in K$, the collection $\{p_{k_0}(B) : B \in \mathcal{B}\} \subseteq \mathcal{A}_{k_0}$ is f.i. On the other hand $X_{k_0} = \ _{B\in\mathcal{B}} p_{k_0}(B)$, because $\ _{k\in K} X_k = \ _{B\in\mathcal{B}} B$, so $\{p_{k_0}(B) : B \in \mathcal{B}\} \subseteq \mathcal{A}_{k_0}$ is non-inadmissible and f.i. Conversely, if for some $k \in K$, $\mathcal{A}_k$ contains a non-inadmissible and f.i subcollection $\mathcal{B}_k$. From this, and by hypothesis i), $\mathcal{A}$

contains a f.i collection $\mathcal{B} = \{p_k^{-1}(B) : B \in \mathcal{B}_k\}$. Moreover,

$$\mathop{}\limits_{k\in K} X_k = p_k^{-1}(X_k) = p_k^{-1}(\mathop{}\limits_{B\in\mathcal{B}_k} B) = \mathop{}\limits_{B\in\mathcal{B}_k} p_k^{-1}(B),$$

then $\mathcal{B} = \{p_k^{-1}(B) : B \in \mathcal{B}_k\}$ is non-inadmissible. Thus we get the following result.

Theorem 4.6. *Under the hypothesis of Theorem 4.5, we have $\mathcal{A}$ contains a non inadmissible and f.i subcollection if and only if $\mathcal{A}_k$ contains a non inadmissible and f.i subcollection, for some $k \in K$.*

Remark 4.7. *In the above Theorem, if $\mathcal{A}$ is a GT on $\mathop{}\limits_{k\in K} X_k$, then the condition, $\mathcal{A}$ is stable for the union of sets, is unnecessary.*

5. Compactness in Generalized Spaces and some applications

In this section we introduce a new form of generalized compactness with respect to a GT on a set. Hence, we obtain a general framework which allows us to derive in a unified way many recent results, concerning the compactness in a product of generalized topologies, product of γ-compact spaces and product of γ-semi compact spaces.

Császár [8], was considered the following notion as an analogue of the concept of compactness.

Definition 5.1. *Let X be a nonempty set, and $\gamma : \exp X \to \exp X$ be an operation. X is said to be γ-compact space if each cover of X composed of γ-open sets has a finite subcover.*

In the following definition, we give another analogue notion of the concept of compactness.

Definition 5.2. *Let X be a nonempty set, μ be a GT on X, and $\gamma : \exp X \to \exp X$ an operation. X is said to be γ-semi compact space if each cover of X composed of γ-semi open sets has a finite subcover.*

We introduce in the following definition which is more general notion from the two definitions above on compactness.

Definition 5.3. *Let X be a nonempty set and μ be a GT on X. X is said to be μ-compact if every f.i collection of μ-open subsets of X is inadmissible.*

The following characterization constitute a generalized version of the Alexander Lemma, for μ-compactness in the context of GT's.

Theorem 5.4. *Let X be a nonempty set, and $\mu = \mu(\mathcal{B})$ a GT on X having $\mathcal{B}$ for base. Then X is μ-compact if and only if every f.i collection of subsets of $\mathcal{B}$ is inadmissible.*

By Theorem 4.6, we obtain the following generalized version of the Tychonoff Theorem for μ-compactness in the GT's.

Theorem 5.5. *Let $K \neq \emptyset$ be an index set. Suppose that, for each $k \in K$, a GT on $X_k \neq \emptyset$ is given. Suppose that μ is a GT on $X = \ {}_{k\in K} X_k \neq \emptyset$, such that*

(i) $p_k^{-1}(M_k) \in \mu$, *if $M_k \in \mu_k$ and $k \in K$;*
(ii) $p_k(M) \in \mu_k$, *if $M \in \mu$ and $k \in K$.*

Then, X is μ-compact if and only if each X_k is μ_k-compact.

By Theorem 5.5 and Theorem 3.13, we have the following interesting corollary

Corollary 5.6. *Let $K \neq \emptyset$ be an index set. Suppose that, for each $k \in K$, a GT on $X_k \neq \emptyset$ is given. Let $\mu = P_{k\in K}\mu_k$ be the product of the GT's μ_k. Then, X is μ-compact if and only if each X_k is μ_k-compact.*

According to the Lemma 3.6, the collection of the γ-semi open sets is a GT on a set X. Moreover, the Theorem 3.14, shows that collection of the γ-semi open sets in a Cartesian product, satisfy the conditions (i) and (ii) in the Theorem 5.5, if we consider in each of its factor the collection of the γ_k-semi open sets. Thus, we have the next result.

Theorem 5.7. *Let $K \neq \emptyset$ be an index set. Suppose that, for each $k \in K$, a GT on $X_k \neq \emptyset$ and the operation $\gamma_k : \exp X_k \to \exp X_k$ on X_k associated to μ_k are given. Suppose that $\gamma : \exp X \to \exp X$ is an operation on $X = \ {}_{k\in K} X_k$, associated to $\mu = P_{k\in K}\mu_k$ and compatible with $\{\gamma_k\}_{k\in K}$, such that $\gamma(\emptyset) = \emptyset$. Then, X is γ-semi compact if and only if each X_k is γ_k-semi compact.*

In the above theorem, if we choose the operations γ and γ_k in adequate form, we obtain many different forms of generalized compactness. The following special cases have been introduced in the literature (see [8],[9]). When γ is taken as $c_\mu i_\mu$, (respectively $i_\mu c_\mu$,$i_\mu c_\mu i_\mu$,$c_\mu i_\mu c_\mu$), the corresponding notion of γ-semi compactness is semi-compactness, (respectively strongly compactness, α-compactness, β-compactness).

It is easy to see that if $\gamma : \exp X \to \exp X$ is an operation on $X = \ {}_{k\in K} X_k$, associated to $\mu = P_{k\in K}\mu_k$ and compatible with $\{\gamma_k\}_{k\in K}$, such that $\gamma(\emptyset) = \emptyset$. Then $\gamma i_\mu : \exp X \to \exp X$ is an operation on $X = \ {}_{k\in K} X_k$, associated to $\mu = P_{k\in K}\mu_k$ and compatible with $\{\gamma_k i_k\}_{k\in K}$, also $\gamma i_\mu(\emptyset) = \emptyset$. So, by using Theorem 3.5, we have the following result.

Corollary 5.8.

(i) A nonempty product space is semi compact iff each factor space is semi compact;
(ii) A nonempty product space is α-compact iff each factor space is α-compact.

Ganster, Janković and Reilly obtain (see [14]) that a topological space (X, τ) is semi-compact if and only if (X, τ_α) is hereditarly compact, using the notion of hereditarly compact introduced by Stone (see [22]). By this result and the previous Corollary, we obtain a corollary as follows.

Corollary 5.9. *A nonempty product space is hereditarly compact if and only if each factor space is hereditarly compact.*

Remark 5.10. *The notions of semi-compactness, α-compactness and hereditarly compact have been studied by many mathematicians in topological spaces (particular case of a GT). In the literature a few results about product of these notions are known.*

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HOJA DE METADATOS

Hoja de Metadatos para Tesis y Trabajos de Ascenso – 1/5

Título	**Sobre Generalizaciones de la Teoría de Fredholm, Teoremas Tipo Weyl, Estructuras Minimales y Topologías Generalizadas**

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	Topología

Resumen (abstract):

Se estudia una generalización de la Teoría de Fredholm, en el sentido de Berkani, y se dan aplicaciones de esta a los teoremas de Weyl. Además se generalizan muchos teoremas clásicos de topología general empleando las nociones de estructura minimal y topología generalizada.

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